--- Commentary ---

A critical understanding of the fractal model of metabolic scaling

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Summary

The exponent of the scaling of metabolic rate with body mass has been the subject of debate for more than a century. The argument is at two levels, one concerning questions of empirical support for the exponent and the other, how to derive it theoretically. At this second level, the exponent is usually treated as the outcome of an underlying physical burden and approached as the search for a natural law emerging within energetic and geometric constraints. Recently, a model relying on fractal geometry was proposed as a general explanation for the phenomenon. In the present study, a reanalysis of the fractal model is performed to verify its validity. All the conditions that allow for the connection between the geometric proposition and the allometric exponent are evaluated, as well as the energy loss minimization procedure put forward in the model. It is demonstrated that the minimization procedure is mathematically incorrect and ill-posed. Also, it is shown that none of the connecting conditions are fulfilled. Therefore, it is concluded that the fractal model lacks self-consistency and correct statement: it relies on strong assumptions of homogeneity in morpho-physiological features among organisms instead of demonstrating them, as claimed by its authors. It is proposed that empiricists and theoreticians should rather evaluate the frameworks for addressing metabolic scaling phenomena.

Key words: allometry, fractal geometry, optimization, metabolic rate, body mass.

Introduction

The problem of the scaling between size and energy metabolism began with the verification, in the late 1800s by Rubner (Hoppeler and Weibel, 2005), that standard metabolic rate does not scale linearly with body mass. This initial evidence came from mammals and the associated exponent of the relationship was 0.67. Subsequently, three authors examined larger data sets and contended that the exponent was 0.75 instead of 0.67 (Kleiber, 1932; Kleiber, 1961) [for Brody in 1945 and Hemmingsen in 1960, see Calder III (Calder, III, 1996), for reference]. From that time to the present day, a series of debates has taken place in the literature concerning the value of and the putative explanation for such an exponent.

One can recognize two levels of scientific dispute in this issue. The first level is, broadly, the question of 'the empirical support to the exponent'. It is related to statistical and data collection matters. To this level of research pertain questions of the type 'do empirical results reassure the model?', 'do the modeled systems fulfill the premises of the model?' and 'how robust is the model in the face of empirical deviations from the predicted?'.

The second level of the dispute is the question of 'how to theoretically derive the exponent'. It is related to the adequate choice of parameters and variables that should be taken into account in the modeling itself. At this level we find questions such as: (1) 'Is the model self-consistent?' and (2) 'Is the model correctly stated?'.

While at the first level of the argument, the value and even the existence of a characteristic allometric exponent is discussed (e.g. Dodds et al., 2001; Heusner, 1984; McKechnie and Wolf, 2004; McNab, 1983; Riisgård, 1998; Suarez and Darveau, 2005; Symonds and Elgar, 2002; White and Seymour, 2003; Weibel and Hoppeler, 2005; Wieser, 1984), the second level begins with the assumption that such an exponent is the outcome of a physical burden. Therefore, studies concerning the latter try to demonstrate that a given value of the allometric exponent, usually 0.75, arises naturally from energy minimization principles under geometrical restrictions. One can find examples in the literature discussing elastic energy scale (McMahon, 1973); similarity principles (Günther, 1975); heterogeneous catalytic bioreactor (Sernetz et al., 1985); constructal law (Bejan, 2000); similitude in

cardiovascular systems (Dawson, 2001); central source and distribution of sinks (Dreyer, 2001); and a fluid dynamics approach (Rau, 2002; Santillan, 2003). From such a viewpoint, the observed relationship can be taken as a phenomenological law, which has been designated the '3/4 law' of biological scaling.

In recent years, attention to this scaling exponent has grown, largely due to the publication of a so-called 'general explanation to the allometric scaling' (West et al., 1997). The explanation is envisaged as a model relying on fractal geometry associated with energy minimization in organismic flow of materials. From this model, the authors suggest that the 3/4 law is straightway derived, thus explaining the empirical observations in '*almost all living beings*' (West et al., 1997). Thus, a natural subject to be addressed concerns the validity of the fractal geometry model.

This study intends to provide answers to questions (1) and (2) posed above, by re-analyzing conceptual issues related to the proposal of the fractal model. For the sake of simplicity, I use WBE for West et al. (West et al., 1997); and I refer to their equations as 'WBEeq. i', where i is the number they have in the original text. So, WBEeq. 4 should read as 'equation number 4 in West et al., 1997'. To facilitate the appraisal, I follow the nomenclature and symbolism of WBE and name the equations in the present study as Eqn i). Thus, Eqn 3 should be understood as the third equation appearing in this study. A list of symbols, with the same nomenclature as in WBE, is provided to facilitate the reading.

The logical structure of the fractal model
General résumé

The aim of WBE is to prove, based on an energy minimization procedure, that the volume of circulating fluid is linearly related to body mass and that a certain geometric structure containing such a fluid is a natural consequence in living beings. Then, with these results, they expect to obtain the value of the scaling exponent of the relationship between resting metabolic rate and body mass.

A detailed view

The fractal model lies on three premises: (i) fractal geometric structure, (ii) terminal units of fixed size and (iii) minimization in energy supplying demands. The authors attach, to this set of premises, five other statements, regarding the fluid mass conservation along the 'circulatory systems', the linear relationship between fluid flow and metabolic rate, the power relationship of metabolic rate and body mass (this is the scaling law), the pulsatile (or not) feature of flows, and the relation of power input with flow and resistances. Finally, the effect of body mass completes this causal core of the model.

Combining elements from such a core, some consequences are then delineated. These comprise the number N of branches a system should have; the volume of the terminal units; the total volume V_b of fluid in the network; and the volume supplied by each terminal unit (the 'service volume', as WBE named it). The final step is to combine elements of this set of consequences among themselves in a simplified manner to obtain the allometric exponent a.

It is possible, thus, to identify the essential conditions that WBE should provide to derive this simplified way to compute a. These are what I name as 'Secondary Independent Consequences', SIC. From the premise of the fractal geometric structure of the system, WBE should show that the ratio γ of the length between daughter and parent tubes has a constant value (SIC 1). From energy minimization in fluid transporting systems, the authors should be able to prove that: the number n of branches arising at each level in the system, called the branching factor or ratio, is constant (SIC 2); the ratio β of the radii between daughter and parent tubes has a constant value (SIC 3); and the fluid volume V_b is linearly related to body mass M_b (SIC 4). Fig. 1 presents the scheme of these logical relationships from the causal core of the model to the final outcome, the allometric exponent a.

The development of the fractal model
(1) Is the model self-consistent?
Impedances and resistances to flow

Fluid flows occur in the face of opposing forces to such flows. These forces arise from the viscous properties of the moving fluids, the geometric structure of the tubing, and the physical features of the pumps/walls of the systems. In metazoans in general, and vertebrates in particular, the major source of force dropping in circulatory systems occurs in small diameter arteries and arterioles (e.g. Milnor, 1990), where flows are of non-pulsatile nature and resistance appears, mainly, as in ohmic circuits. This results in the well-known Hagen–Poiseuille relation among flow, radius, length and pressure drop along the tube. Besides this major source of energy dissipation, the cyclic nature of cardiac enthalpy output imposes another energy sink in circulatory systems. This appears as the wave propagation in the walls of large arteries, generating an impedance to flow. Finally, in non-convective flow regimens, as in diffusion-driven processes, the limitation to exchange due to the geometric arrangement of the conducting tubes can also be regarded as a resistance to the flow, and the disposition of these resistances would also be related to the energy demands of the organism.

Impedance matching is the conceptual framework in which to address the issue. It can be put simply as the way to combine resistances (impedances) such that there would be no 'excess' in any point of the system (and, thus, no 'shortage' either). Therefore, WBE intended to solve this problem through two flow regimens.

(i) Pulsatile flows

To approach energy minimization in pulsatile flows, the fractal model takes into account the impedance Z in these flow regimens, related to the wave propagation in vessel walls. The usual approach to describe wave propagation is to solve a
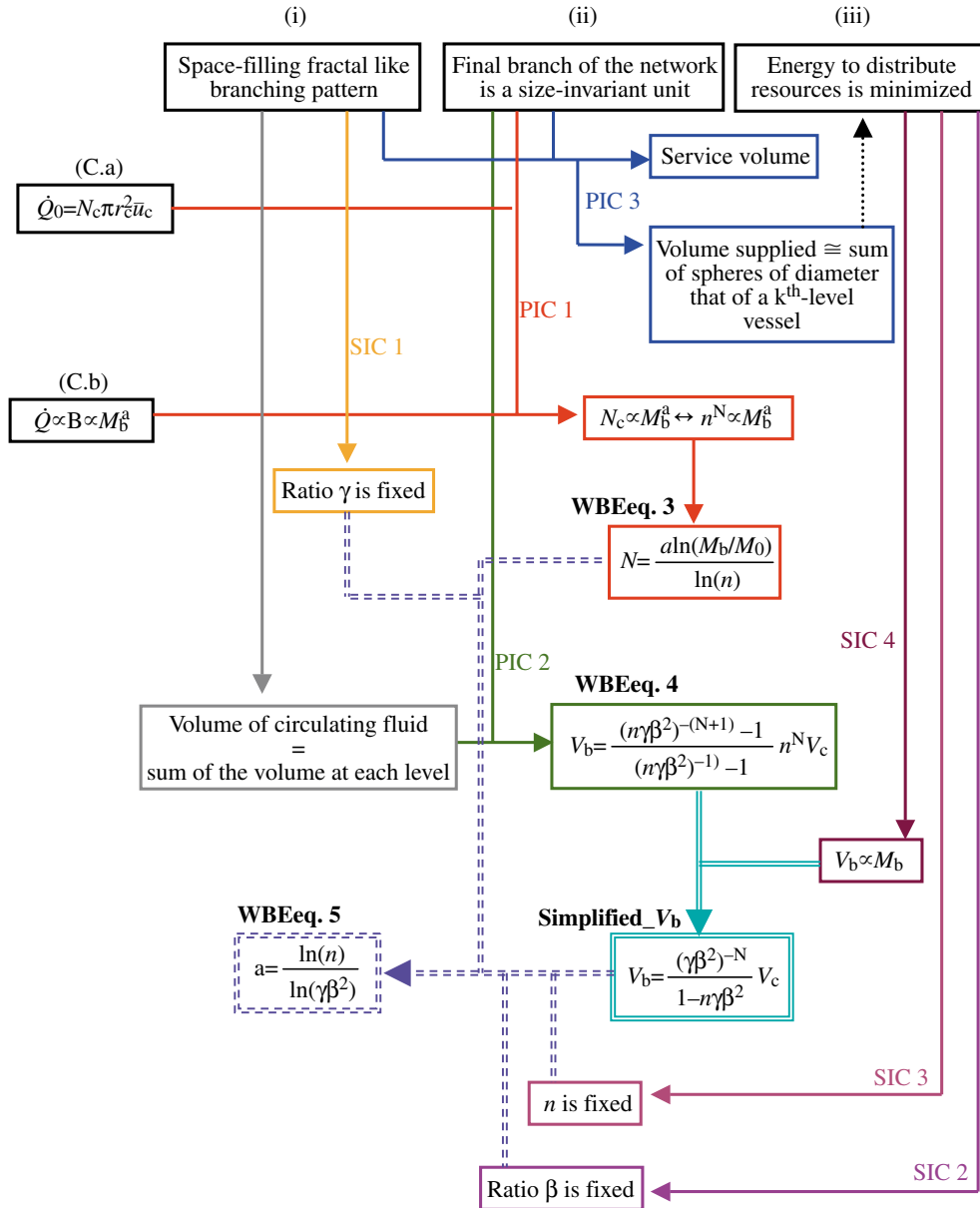
Fig. 1. Schematic view of the logical structure of the fractal model. Black boxes contain the causal core of the model, namely its three premises (i–iii), mass conservation (C.a) and scaling (C.b). The three primary independent consequences (PIC) WBE obtain by combining elements from the causal core are shown in red, green and blue boxes. Arrows 'SIC 1' to 'SIC 4' indicate the secondary independent consequences WBE need to prove from the core. Then, another step of combining boxes is done by WBE to obtain a simplified equation of the circulating fluid from WBEeq. 4. A final round of merging PICs and SICs boxes is needed to obtain the putative scaling exponent *a*, in WBEeq. 5. Notice, also, that PIC 3 're-enters' the causal core in the energy minimization proposed by WBE, by means of a 'volume restriction' term. This is discussed in section 2.

partial differential equation of position/velocity in time and space (e.g. Boyce and Diprima, 2000). By doing this, the authors obtain (WBEeq. 8)[1]:

$$Z \approx \frac{c_{0,i}^2 \rho}{\pi r_i^2 c_i}, \tag{1}$$

and conclude that for high Womersley numbers, such as those found at the aorta level, $c = c_0$ so that 'the *r* dependence of *Z* has changed from the nonpulsatile r^{-4} to r^{-2}.' (WBE). Noting that c_0 is:

$$c_0 \equiv \sqrt{\frac{E_i h_i}{2 \rho r_i}} \tag{2}$$

and applying Eqn 2 into Eqn 1, results in a *Z* dependence on *r*

[1]Eqn 1 reads a little differently from WBEeq. 8 because I have highlighted that *r*, *c* and c_0 refer to the level 'i' of the branching system.

raised to –2.5. However, to compute a=0.75 from the fractal model, the authors must obtain the ratio between radii $\beta=n^{-1/2}$ (i.e. r raised to –2). Matching impedances gives:

$$Z_{i+1} = \frac{\sqrt{\rho}}{\pi\sqrt{2}} \left(\frac{\sqrt{E_{i+1}h_{i+1}}}{r_{i+1}^{2.5}} \right) = \frac{1}{n_i} \left(\frac{\sqrt{\rho}}{\pi\sqrt{2}} \right) \left(\frac{\sqrt{E_i h_i}}{r_i^{2.5}} \right) = Z_i . \quad (3)$$

Re-arranging Eqn 3:

$$\beta_i = \left(\sqrt{\frac{E_i h_i}{E_{i+1}h_{i+1}}} \right)^{1/2.5} n_i^{-1/2.5} . \quad (4)$$

From Eqn 4, one can observe that for the desired condition $\beta=n^{-1/2}$ to be fulfilled a key relationship is needed: the ratio $(E_i h_i)/(E_{i+1}h_{i+1})$ must be a fixed value ($=n^{-1/2}$) independent of the level of the branching system. In other words, one must expect to find a fixed relationship among elastic modulus, wall thickness and radius along the branching system. This is far from being supported by empirical evidence. In fact, empirical evidence is much more favorable to the '$n^{-1/2.5}$ rule' instead of the '$n^{-1/2}$' (see Huang et al., 1996); but this discussion, despite of its relevance, is not within the scope of the present analysis.

WBE put forward the fractal model by omitting the elastic modulus challenge and stating that energy minimization loss gives h_i/r_i constant independent of the level i. Inspection of Eqn 3 shows that this requirement is not true: impedance matching is obtained through several different combinations among elastic modulus, wall thickness and radius. Consequently, the fractal model relies on the unproven relationship $\beta=n^{-1/2}$ of radii ratio for pulsatile flow regimens.

(ii) Non-pulsatile non-cyclic flows

The matching principle that WBE employ to approach non-pulsatile non-cyclic flows comes from the relationship $\beta=n^{-1/2}$ arising from the 'rigid-pipe model', assuming an extremely low velocity of the circulating fluid, as in a diffusion-driven process. Such an impedance matching excludes the length of the tubes from the problem. This is because the matching is related only to the area: one tube with a given cross-sectional area and length would present the same 'resistance' as two tubes of half an area each and with the same length of the single one. However, flow in sap conduits, despite being very slow, cannot be taken as a diffusive process. Nevertheless, even if one accepts that resistance is related only to $\beta=n^{-1/2}$, and that the length ratio γ relates with the branching ratio n by $\gamma=n^{-1/3}$, the proof that the volume of the circulating fluid, V_b, is linearly related to body mass M_b (SIC 4, see Fig. 1 and the next section) is still missing.

Therefore, the question is when the fractal model for non-pulsatile non-cyclic flows becomes supported by the fundamental relationship $V_b \propto M_b$. Apparently, the proponents approached this problem in WBEeq. 7 (see, also, section 2), an equation obtained when they are considering the 'cardiac output as a function of all relevant variables' (WBE). Consequently, if $V_b \propto M_b$ for non-pulsatile flows in pumpless processes is to be considered under such an umbrella, one

should also consider the relationship $\beta=n^{-1/3}$ obtained there, instead of the $\beta=n^{-1/2}$ just presented. However, this would lead to a scaling exponent different from 0.75, as stated by the authors themselves.

Taken together, the impedance matching of the fractal model for either flow regimens do not lend support to the allometric exponent WBE wish to obtain.

Fluid volume

In this section, we need to address a series of interconnected propositions which, when combined with the energy minimization procedure, supposedly would lead to the necessary deduction that the volume of the circulating fluid is linearly related to body mass, a key step in the model (see Fig. 1). As just shown, WBE have never proved $V_b \propto M_b$ arising from impedance matching for non-pulsatile non-cyclic regimens. Then, the question is whether this linear relationship is obtained for the other flow regimen.

Let us analyze the equation that WBE obtain for computing the volume of circulating fluid under pulsatile flows. This equation is their WBEeq. 9, which the authors put as a generalization of the relationship between the volume of circulating fluid and the volume of the terminal unit [i.e. WBEeq. 4]:

$$V_b = \frac{V_c}{(\beta_>^2 \gamma)^N} \left\{ \left(\frac{\beta_>}{\beta_<} \right)^{2\bar{k}} \left(\frac{1-(n\beta_>^2\gamma)^{\bar{k}}}{1-(n\beta_>^2\gamma)} \right) + \left[\left(\frac{1-(n\beta_>^2\gamma)^N}{1-(n\beta_>^2\gamma)} \right) - \left(\frac{1-(n\beta_>^2\gamma)^{\bar{k}}}{1-(n\beta_>^2\gamma)} \right) \right] \right\} . \quad (5)$$

In the coding, $\beta_<=n^{-1/2}$ and $\beta_>=n^{-1/3}$; $\gamma=n^{-1/3}$. The authors conclude that if the ratio between radii is related to the branching ratio to the –1/3 power, then the allometric exponent a equals to 1. Employing the value of $\beta_>$ to solve Eqn 5[2], it is found that $n(n^{-1/3})^2 n^{-1/3}=nn^{-2/3}n^{-1/3}=nn^{-3/3}=n/n\equiv1$. Thus, there are three cases of $(1-1)/(1-1)$, i.e. 0/0, in the equation:

$$V_b = \frac{V_c}{(\beta_>^2\gamma)^N} \left\{ \left(\frac{\beta_>}{\beta_<} \right)^{2\bar{k}} \left(\frac{1-1}{1-1} \right) + \left[\left(\frac{1-1}{1-1} \right) - \left(\frac{1-1}{1-1} \right) \right] \right\} = \frac{V_c}{(\beta_>^2\gamma)^N} \left\{ \left(\frac{\beta_>}{\beta_<} \right)^{2\bar{k}} \left(\frac{0}{0} \right) + \left[\left(\frac{0}{0} \right) - \left(\frac{0}{0} \right) \right] \right\} . \quad (5a)$$

These meaningless results arise because of the use of the formula for the sum of a power series without taking into account the possibility that the product $n\gamma\beta^2$ could be equal to 1.

The question is how the authors concluded that if $\beta=n^{-1/3}$ then a=1. Considering that, in this case, $\gamma\beta^2=n^{-1}$, and inserting this result directly into WBEeq. 5, one would obtain $a=-\ln(n)/-\ln(n)$, thus a=1, unless one has a 0/0 case.

[2]Or for any other equation of V_b and Z in WBE, indeed.

Given that WBE stated, as independent consequences of their core of premises, that the total number of branches in a system is proportional to $M_b{}^a$, and that the volume of fluid in the network is obtained as a sum of the volume contained at each level (these are PIC 1 and PIC 2, see Fig. 1), one may directly combine them. By doing this, it is now possible to derive the volume of circulating fluid in relation to body mass (i.e. to make $\partial V_b/\partial M_b$). Such a procedure would immediately allow one to obtain a relationship between mass and fluid volume, perhaps the putative linear relationship $V_b \propto M_b$ (i.e. SIC 4, see Fig. 1). Note, if there is a linear relationship between two variables, then the derivative results in a constant (this is the slope of a straight line relating the variables). When one takes this direct path and combines WBEeq. 3 with WBEeq. 4 to derive V_b in relation to the independent variable M_b, and set $\partial V_b/\partial M_b$=constant=C, two interesting things emerge.

Firstly, in our case of interest, i.e. when $\gamma\beta^2=n^{-1}$, the condition for $\partial V_b/\partial M_b$=C is that $n=1$. This means that the branching rule of the system is to have no branches, and $a=\ln(1)/\ln(1)\equiv0/0$. Therefore, it is not true that $\beta=n^{-1/3}$ leads to $a=1$.

Secondly, setting $\partial V_b/\partial M_b$=C directly from WBEeq. 3 and WBEeq. 4, and not considering the $\gamma\beta^2=n^{-1}$ cases, it becomes clear that $V_b \propto M_b$ is not a general result as considered by WBE. In fact, by inspection of the simplified version of the fluid volume equation (equation Simplified_V_b in Fig. 1), it can be verified that there is a restricted set of combinations of the product $\gamma\beta^2$ that allows for $V_b \propto M_b$, as follows. Considering that the product $\gamma\beta^2=n^{-s}$ (s is some arbitrary value) and that $n^N \propto M_b{}^a$ (this is PIC 1, see Fig. 1), one obtains by rearranging the numerator in the Simplified_V_b equation:

$$V_b = \frac{(M_b{}^{-a})^{-s}}{1-n^{1-s}} \, V_c \, . \qquad (6)$$

Thus, $\partial V_b/\partial M_b$=C if and only if $a=1/s$. Because WBEeq. 3 and WBEeq. 4 come as independent consequences from the core of the causal factors in the model, the meaning of Eqn 6 is clear: the desired linear relationship between the volume of circulating fluid and body mass (SIC 4, i.e. $V_b \propto M_b$) can be obtained by the energy minimization procedure if and only if one knows beforehand that $a=1/s$. In other words, the allometric exponent a cannot be fairly obtained by means of the alleged energy minimization procedure because prior knowledge of the value of such an exponent is required.

The bottom line is that WBE cannot prove their claim that a linear relationship between fluid volume and body mass is a natural consequence of geometric/impedance constraints in living beings.

As a final comment in this sub-section, it is interesting and intriguing to query why WBE did not take the direct and obvious step described above to approach what their model predicts about the relationship between circulating fluid and body mass. Another option they had would be to impose $V_b \propto M_b$ and obtain the value of a. Obviously, this latter procedure would banish the flavor and the appeal of an energy minimization procedure.

Service volume

The next point to be addressed is whether the service volume put forward by WBE is in accordance with both their energy minimization principle and the scaling rule. From the premises of the model, WBE state that a service volume is a group of cells supplied by each invariant terminal vascular unit (capillary) and that each service volume is a sphere with a volume defined by the invariant length of the terminal vascular unit (see PIC 3 in Fig. 1). Such a volume is computed as $4/3\pi(l_c/2)^3$. Thus, it is imposed that the service volume should be an invariant unit as a consequence of the model itself. Because a service volume as defined by WBE is an invariant unit, the established relationship should have the form of:

$$\sum_1^{N_c} \frac{4}{3} \, \pi \left(\frac{l_c}{2}\right)^3 = N_c \frac{4}{3} \, \pi \left(\frac{l_c}{2}\right)^3 = \frac{M_b}{\rho_o} \, , \qquad (7)$$

where ρ_o is the body density of the organisms throughout the lineage under study. Eqn 7 says, simply, that the total volume serviced by the network, i.e. the volume of the organism itself, is linearly related to the mass of such an organism, unless density varies with body mass. However, it is stated that $N_c \propto M_b{}^a$. Thus, the volume serviced by the network cannot be linearly related to the body mass of the organism (unless $a=1$). It is interesting and relevant that Kozlowski and Konarzewski arrived at the same conclusion by means of a different reasoning (Kozlowski and Konarzewski, 2004).

At this point, one realizes another serious inconsistence of the fractal model. Because the service volume scales with $M_b^{1/4}$, according to the fractal model, the bigger the organism the bigger the volume serviced by each terminal unit. From the causal core of the model, the proponents state that the mean velocity of fluid in the terminal units and the pressure drop along such terminal units are constants independent of size. Because entropy generation minimization (and so, minimization of energy loss) is required to minimize the pressure drop ΔP (e.g. Glansdorff and Prigogine, 1971; Bejan, 1996), the fractal model leads to the conclusion that small organisms are not operating to fulfill the energy loss minimization expected or, alternatively, big organisms operate well below the predicted minimum. The bottom line is that the model cannot resolve the 'size demand' and the 'energetic demand' at the same time.

(2) Is the model correctly stated?

This last section is dedicated to an analysis of the computations of energy minimization proposed by WBE. The power W emerging as external work to drive flow $\dot{Q}$ facing the impedance Z is given by $W=\dot{Q}^2 Z$. The path that WBE took for their approach was to impose some restrictions in the process (see below) and then to minimize W by means of Lagrange

multipliers. In order to do this, they need to construct what is known as an 'augmented function', F. This augmented function is constructed as a sum of the original function (i.e. W above) with the product of each restriction by a Lagrange multiplier, λ_j. The use of Lagrange multipliers is one of several ways of solving an optimization problem, and there are some specific mathematical impositions that ought to be fulfilled in order to have a well-stated solvable problem [see (Rockafellar, 1993) for a deeper discussion on the theme]. WBEeq. 7 for the optimization procedure is reproduced below:

$$F(r_k,l_k,n) = W(r_k,l_k,n_k,M_b) +$$
$$\lambda V_b(r_k,l_k,n_k,M_b) + \sum_{k=0}^{N} \lambda_k N_k l_k^3 + \lambda_M M_b \; . \quad (8)$$

One of the primary mathematical impositions to solve an optimization problem by Lagrange multipliers is related to the number of variables and restrictions in the function. If the number of variables is 3 (i.e. r_k, l_k, n_k, in this case), then the rank of the matrix of the restrictions must be lower than 3. Putting it simply, there could be at most two restrictions in this problem. Inspection of Eqn 8 shows that the restrictions easily outnumber three: there are $N+3$ restrictions there. The end result here is that either the problem becomes solved beforehand, irrespective of the optimization of the performance imposed, or it is unsolvable. Notice, also, the dilemmas caused by WBE in writing $W(r_k,l_k,n_k,M_b)$, but stating $F(r_k,l_k,n)$: (1) n was not yet proven as constant at this point, thus it ought to be n_k; (2) M_b is treated as a variable for W but as a constraint to the augmented function.

A constraint in an optimization problem is some variable that should be treated as a parameter or a fixed value, and it must be a function of the variables in the problem. For example, one would try to maximize the area of a given polygon subjected to the constraint of a certain perimeter. The perimeter is, in that case, a fixed value and it can be written as a function of the variables determining the area. The question is, then, what is meant by the restrictions in WBE modeling. It is completely unknown how the restriction 'mass' is to be written as a function of r_k, l_k and n_k, among other reasons because M_b is presented as the leading factor in the scaling phenomenon under study, indeed.

The second term on the right-side of Eq. 8 is the 'fluid restriction'. The fluid volume was not taken into account as part of the energetic demand of the system and it is the dependent variable that WBE are looking for to complete SIC 4. In other words, it is not possible to define the value of V_b to be taken as a constraint.

The third term in the right-side is

$$\sum_{k=0}^{N} \lambda_k N_k l_k^3 \; .$$

The closest entry in WBE to this term is the volume-preserving fractal (PIC 3), where the volume is given by

$$N_k \frac{4}{3} \pi \left(\frac{l_k}{2} \right)^3$$

(see Eqn 7). Thus, apparently, this term is what the authors mean by 'subject to a space-filling geometry'. However, now the geometry is no longer taken as the proposed spheres. Simply, it has become a cube (or a sum of cubes), without any clear explanation.

In fact, all the constraints in WBEeq. 7 (Eqn 8) are ill-posed because they all mean that the 'restriction' under concern is equal to zero: when posing an optimization problem *via* Lagrange multipliers each λ_j multiplies a constraint in the form $f_j=0$. Thus, the fractal model is developed over the following statements:

$$V_b = 0 \; ,$$
$$N_k l_k^3 = 0 \text{ for all } k \; ,$$
$$M_b = 0 \; .$$

A logical set of constraints should have the form:

$$V_b - \Phi_b = 0 \; ,$$
$$N_k l_k^3 - \Phi_N = 0 \; ,$$
$$M_b - \Phi_M = 0$$

where the Φ are real-value functions related to size. However, even with the aid of this new set of constraints, the proposition of the problem remains tautological: the function Φ_M is related to size, which is determined by body mass; and, obviously, since Φ_b must be known at this time, then the relationship between fluid volume and body mass must also be already known.

In the following paragraphs, the minimization procedure proposed by WBE (i.e. Eqn 8) for the case of Hagen–Poiseuille flow is analyzed. As Fig. 1 highlights, to develop such a procedure is crucial to understanding what can fairly be obtained by the fractal model.

WBE propose that minimizing the power W can be treated simply as minimizing the impedance Z. Therefore, the internal functions in F are:

$$Z_T = \sum_{0}^{N} Z_i = \sum_{0}^{N} \frac{8\mu l_i}{\pi r_i^4 N_i} \; , \quad (9)$$

$$V_b = \sum_{0}^{N} \pi r_i^2 l_i N_i \; , \quad (10)$$

$$A_N = \sum_{0}^{N} \lambda_i N_i l_i^3 \; . \quad (11)$$

Notice that I use N_i for the number of vessels at level i instead

of 'n^i' as WBE did in their WBEeq. 4 and WBEeq. 6. This is because WBE proposed to prove that n is a fixed value in the energy minimization procedure (see Fig. 1), something yet to be done. The derivatives of F in relation to r_k, l_k and n_k are clearly shown elsewhere (Dodds et al., 2001), so I omit them here (they are presented in the Appendix). However, it is important to keep track of the consequence that they have: because of the geometric assumptions that γ_k is a fixed value and that

$$N_k \frac{4}{3} \pi \left(\frac{l_k}{2}\right)^3$$

is the same for all levels, it is possible for WBE to obtain both n_k and β_k as a fixed value, but this does not come from the energy minimization procedure. It is simply a consequence of the geometric impositions taken for granted in the fractal model.

Notice that, up to this point, the linear relationship of circulating fluid with body mass (SIC 4) has not yet been demonstrated. By taking the derivatives of F in relation to Lagrange multipliers, it would be expected to obtain this crucial step $V_b \propto M_b$.

(1) $\partial F/\partial \lambda = 0$:

$$\frac{\partial(\lambda V_b)}{\partial \lambda} = 0 \Leftrightarrow V_b = 0 \,,$$

which is the restriction V_b proposed by WBE (i.e. the volume of circulating fluid is zero).

(2) $\partial F/\partial \lambda_i = 0$:

$$\frac{\partial(\lambda_i N_i l_i^3)}{\partial \lambda_i} = 0 \Leftrightarrow N_i l_i^3 = 0 \text{ for all i} \,,$$

which is the 'volume-preserving' restriction proposed by WBE, thus implying that either the number of branches at level i is zero ($N_i=0$) or that the length of the vessels at that level i is zero ($l_i=0$). And this occurs for all levels i, as expected from the constraints.

(3) $\partial F/\partial \lambda_{Mb} = 0$:

$$\frac{\partial(\lambda_{Mb} M_b)}{\partial \lambda_{Mb}} = 0 \Leftrightarrow M_b = 0 \,,$$

which is the 'mass' restriction proposed by WBE. As can be seen, the fractal model works on the result of mass equal to zero for all the range of body sizes.

The reader should be aware that the optimization problem stated by WBE has just ended: because body mass was considered as a restriction, as extensively discussed above, there are no further steps. The derivatives of F are taken in relation to the variables and Lagrange multipliers, not in relation to the restrictions. However,

unexpectedly and incorrectly, WBE proceeded and made $\partial F/\partial M_b$.

Were such a step a correct one, it would be logical to derive F in relation of both V_b and also 'the volume preserving term'. Were these steps taken, then none of the results conceded by Dodds et al. (see above) would remain valid (Dodds et al., 2001): all λ and λ_k would be found equal to zero (see Appendix).

However, it would be enlightening to proceed and take the incorrect step $\partial F/\partial M_b$. Writing down $\partial F/\partial M_b=0$ directly from Eqn 12), WBE would expect to obtain:

$$\frac{\partial(Z_T)}{\partial M_b} + \frac{\partial(\lambda V_b)}{\partial M_b} + \frac{\partial(A_N)}{\partial M_b} + \frac{\partial(\lambda_{Mb} M_b)}{\partial M_b} = 0 \Leftrightarrow$$

$$\frac{\partial(Z_T)}{\partial M_b} + \lambda \frac{\partial(V_b)}{\partial M_b} + \frac{\partial(A_N)}{\partial M_b} + \lambda_{Mb} = 0 \,. \quad (12)$$

Consider, first, the term $\partial(A_N)/\partial M_b$. This term is a sum from the level 0 to the level N of the total levels of branches (see above), and N is dependent on body mass (WBEeq. 3). Therefore, $\partial(A_N)/\partial M_b$ requires much more than simply the derivation of the internal summand: it is imperative to take into account the variation in the upper limit of the sum as well. It would be by no means a trivial task to map the continuous function M_b into the discrete upper limit N in the summation term.

Once again, let us disregard this problem, and try to continue with the $\partial F/\partial M_b$ step. To obtain $V_b \propto M_b$, Eqn 12 should result in:

$$\frac{\partial V_b}{\partial M_b} = -\frac{1}{\lambda}\left(\frac{\partial Z_T}{\partial M_b} + \frac{\partial A_N}{\partial M_b} + \lambda_{Mb}\right) = \text{constant} \,, \quad (13)$$

which implies:

$$\frac{\partial Z_T}{\partial M_b} + \frac{\partial A_N}{\partial M_b} = \text{constant} \,. \quad (14)$$

One can now understand the *raison d'être* of that unexpected term describing what the authors called 'volume restriction', i.e. the sum of cubes

$$\sum_{k=0}^{N} \lambda_k N_k l_k^3 \,.$$

If the result for λ_k from Eqn A3 in the Appendix is applied into the A_N term, the terms Z_T and A_N cancel each other and then:

$$\frac{\partial V_b}{\partial M_b} = -\frac{\lambda_{M_b}}{\lambda} = \text{constant} \,,$$

the general, but erroneous, result WBE wished to obtain.

Such a result occurs merely because of a series of equivocated procedures in the optimization problem put forward in the model. Notice that the so-called 'volume restriction', which really means volume=0 (see above), was

constructed in a way to cancel the impedance in non-pulsatile flows. Furthermore, considering the impedance in pulsatile flows, the forced equality in Eqn 14 vanishes once and for all. In other words, in the case of pulsatile flows, it is impossible to satisfy the incorrect derivation $\partial F/\partial M_b$ to obtain $\partial V_b/\partial M_b$=constant even with the aid of the artificial 'volume restriction' term.

Recently, a model that supposedly resolves the elastic modulo inconsistency found in WBE was proposed (Barbosa et al., 2006). However, it also incurs the same set of mistakes in the energy minimization procedure analyzed here.

We are in a position, then, to conclude that WBE resolved an unrealistic problem in which metabolic rate neither varies nor seeks to be minimized across the phylogenetic tree. In fact, not even body mass is allowed to change in the fractal model. When WBE made their optimization procedure, they considered that cardiac output was adequate '*to sustain a given metabolic rate in an organism of fixed mass M* …'. From that condition, they assumed that W in WBEeq. 7 could be replaced by Z and '*this problem is tantamount to minimizing the impedance Z* …'. However, when variations in mass are taken into account, the model statement becomes significantly different. Considering such variations, cardiac output should be written as:

$$W = \dot{Q}_0^2 Z = \varphi B^2 Z = \varphi B_0^2 M_b^{2a} Z(M_b) \, ,$$

where B is basal metabolic rate; $Z(M_b)$ indicates that impedance is also a function of body mass, and, then, the correct term $\partial W/\partial M_b$ is[3]:

$$\frac{\partial W}{\partial M_b} = \varphi B_0^2 \left(2a M_b^{2a-1} Z + \frac{\partial Z}{\partial M_b} M_b^{2a} \right) .$$

Consequently, the problem cannot be summarized by 'minimizing the impedance'. The real problem is to jointly minimize W and B to variations in body mass.

Discussion

Scaling phenomena are fascinating whenever they appear, and this is not merely because these phenomena can be represented somehow easily by means of a mathematical formulation. As pointed out (Barenblatt, 2003), scaling laws reveal a crucial feature of the phenomenon, namely, its self-similarity. Thus, the search for a scaling law deserves a significant scientific effort because it would render a large amount of data, many times disparate at a first glance, into a concise set of relationships.

Despite many attempts to propose the law underlying the resting metabolic rate *versus* body mass scaling, the

[3]This equation can be taken as the formal demonstration that the allometric exponent must to be known beforehand in order to truthfully compute fluid volume V_b in the energy minimization procedure proposed by WBE: here we see that if the exponent *a* is unknown at this point, then there is the need of another equation to solve the system.

explanations for the desired goal turned out to be elusive. Because contemporary biological phenomena are not history-free, the phylogenetic relationships have to become part of the analysis. Subtle changes in grouping may lead to significant changes in the exponent of a putative scaling exponent and/or change the possible set of causative explanations. For example, if one accepts that food habits are to be considered as a grouping factor (e.g. McNab, 1983), both the exponent and the explanatory set of causes of the relationship change.

Consequently, there is a huge difficulty behind the scenes of this research program. In order to obtain a scaling law, an intermediate asymptotics approach must be taken (Barenblatt, 2003). This simply means that one should be able to recognize the leading terms governing a given phenomenon and leave aside all the peripheral details particular to each time the phenomenon is observed. From such a viewpoint, it is easy for a biologist to identify what generates most of the disagreement in the issue, which are, ultimately, the evolutionary 'details' that are to be put aside.

In trying to state a scaling law, theoretical studies assume a core of properties of the biological systems under analysis, correctly performing the intermediate asymptotics step. In the case of resting metabolic rate, these studies hold, in such a core, that organisms maintain their energy expenditure as a 'single purpose' optimized machine. However, this is not correct. Because of their evolutionary history and habitats, living beings are optimized for 'multi purposes' and resting metabolic rates are the end result of these multiple processes (e.g. Glazier, 2005; Gomes et al., 2004; Hochachka et al., 2003; Lovegrove, 2003; Munoz-Garcia and Williams, 2005; White and Seymour, 2004). It is not surprising, then, that maximum metabolic rate, as in exercise or conditions of cold exposure, for example, turns out as a much better phenomenon to be addressed in scaling studies. Under such extremes, organisms are operating close to a sole purpose process (e.g. Weibel and Hoppeler, 2005).

The work of West et al. (West et al., 1997) had an importance that cannot be denied since it sparked the fuel of the metabolic scaling research program once again. However, the present study shows that the development of the fractal model suffers from various mistakes, and ultimately that the model statement is incorrect in its essential part: the energy minimization procedure. Therefore, the authors put forward a geometric structure from which they can obtain a scaling exponent only if provisos of linear scaling of fluid and of a regular fractal are forced. In addition, the debate over the 'single purpose' *versus* 'multi purpose' system is oversimplified in the fractal model. The authors seek a solution minimizing only the power expenditure for convective transport while a more realistic real problem should lie in the joint minimization of power demand for both the convective transport and the system itself.

Therefore, the quest for a theoretical explanation of the scaling law of resting metabolic rate, if such a law exists at all, remains open.

Appendix

Optimization procedure: the derivatives of F in relation to l_i, r_i and N_i

(1) $\partial F/\partial l_i = 0$:

$$\frac{\partial Z_T}{\partial l_i} + \frac{\partial \lambda V_b}{\partial l_i} + \frac{\partial A_N}{\partial l_i} = 0 \Leftrightarrow \frac{8\mu}{\pi r_i^4 N_i} + \lambda \pi r_i^2 N_i +$$

$$3\lambda_k N_i l_i^2 = 0 \Leftrightarrow 8\mu + \lambda \pi^2 r_i^6 N_i^2 + 3\pi \lambda_k r_i^4 N_i^2 l_i^2 = 0 . \quad \text{(A1)}$$

(2) $\partial F/\partial r_i = 0$:

$$\frac{\partial Z_T}{\partial r_i} + \frac{\partial \lambda V_b}{\partial r_i} + \frac{\partial A_N}{\partial r_i} = 0 \Leftrightarrow -4\frac{8\mu l_i}{\pi r_i^5 N_i} + 2\lambda \pi r_i l_i N_i = 0 \Leftrightarrow$$

$$2l_i(\lambda \pi^2 r_i^6 N_i^2 - 16\mu) = 0 \Leftrightarrow \lambda \pi^2 r_i^6 N_i^2 = 16\mu . \quad \text{(A2)}$$

Inserting Eqn A2 in Eqn A1:

$$8\mu + 16\mu + 3\pi \lambda_k r_i^4 N_i^2 l_i^2 = 0 \Leftrightarrow \pi r_i^4 N_i^2 l_i^2 = \frac{8\mu}{-\lambda_k} , \quad (\lambda_k < 0) .$$

$$\text{(A3)}$$

(3) $\partial F/\partial N_i = 0$:

$$\frac{\partial Z_T}{\partial N_i} + \frac{\partial \lambda V_b}{\partial N_i} + \frac{\partial A_N}{\partial N_i} = 0 \Leftrightarrow -\frac{8\mu l_i}{\pi r_i^4 N_i^2} + \lambda \pi r_i^2 l_i +$$

$$\lambda_k l_i^3 = 0 \Leftrightarrow -8\mu l_i + \lambda \pi^2 r_i^6 N_i^2 l_i + \lambda_k \pi r_i^4 N_i^2 l_i^3 = 0 . \quad \text{(A4)}$$

Inserting Eqn A2 and Eqn A3 in Eqn A4:

$$-8\mu l_i + 16\mu l_i - \frac{\lambda_k 8\mu l_i}{\lambda_k} = 0 \Leftrightarrow 0\mu l_i = 0 . \quad \text{(A5)}$$

That can be of no use in solving for l_i. As explained in the text, the problem is not well posed. As shown in Dodds et al. (Dodds et al., 2001), by combinations among Eqn A1–A4, because Lagrange multipliers are constants, one can obtain the fixed relationship $\beta = n^{-1/3}$, but only with provisos of a regular geometric structure.

The incorrect derivation of F in relation to constraints of the problem

Direct inspection of WBEeq. 7 reveals that if derivatives of F are taken in relation to V_b and A_N, then:

$$\frac{\partial F}{\partial V_b} = 0 \Leftrightarrow \lambda = 0 ,$$

$$\frac{\partial F}{\partial A_N} = 0 \Leftrightarrow \lambda_k = 0 .$$

Then, the combinations among Eqn A1–A4 are meaningless: from Eqn A1 to Eqn A3, $\mu \equiv 0$, and the solution conceded by Dodds et al. (Dodds et al., 2001) is forbidden.

List of symbols

a	allometric exponent
A_N	$\sum_0^N \lambda_i N_i l_i^3$
B	basal metabolic rate
c	wave velocity in vessel
E	elastic modulus of vessel wall
F(˙)	augmented function in the optimization procedure by Lagrange multipliers
h	thickness of vessel wall
l	length of the vessel
M_b	body mass
M_0	unitary body mass
N	the total number of branching levels in a given organism
n_i	branching ratio (factor) at level i (i.e. the number of daughter branches arising from a parent)
N_i	the number of branches at level i (not to be confused with the single N)
PIC	primary independent consequences
$\dot{Q}$	convective flow within the vessels of an organism
r	radius of the vessel
SIC	secondary independent consequences
subscripted 0	primary vessel of the system
subscripted c	capillary (terminal unit)
subscripted i or k	level at the branching system
subscripted T	total
$\bar{u}_c$	mean velocity in capillaries
V_b	fluid volume within the vessels
V_c	single capillary vessel volume
W	power output (cardiac output)
WBE	West et al. (1983)
Z	impedance
β_i	the ratio r_{i+1}/r_i
Φ	real-valued function related to size
γ_i	the ratio l_{i+1}/l_i
λ	Lagrange multiplier of the restriction V_b
λ_k	Lagrange multiplier of each restriction $N_i l_i^3$
λ_{Mb}	Lagrange multiplier of the restriction M_b
μ	viscosity

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