NOTE ON THE CALCULATION OF PROPELLER EFFICIENCY USING ELONGATED BODY THEORY

JIAN-YU CHENG AND REINHARD BLICKHAN*†

A. G. Nachtigall, Fachberich Biologie/Zoologie, Universität des Saarlandes, D-66123 Saarbrücken, Germany

Summary

The elongated body theory has been widely used for calculations of the hydrodynamic propulsive performance of swimming fish. In the biological literature, terms containing the slope of the amplitude function at the tail end have been neglected in the calculations of thrust and efficiency, and a slope of zero has been assumed. However, some fishes, such as saithe and trout, have non-zero values of the slope near the tail end and, when this term is taken into account, the efficiency may be reduced by as much as 20% and approaches the result given by the three-dimensional waving plate theory. The inclusion of the slope in the efficiency considerations results in an optimum ratio of the swimming speed to the wave speed that is clearly less than 1. It is suggested that the slope terms should be included in the estimation of propulsive performance for fish swimming with variable amplitude.

Introduction

The propulsive efficiency of swimming motion is a very important variable in the energetics of aquatic animal locomotion. In many studies of fish swimming, a simple formula for hydrodynamic propulsive efficiency (η) based on the elongated body theory (Lighthill, 1960, 1975) has been employed:

$$\eta = \frac{1}{2} \left[1 + \frac{U}{V} \right] , \tag{1}$$

where U is the swimming speed and V is the wave speed of undulation (e.g. Webb, 1975; Videler and Wardle, 1978; Tang and Wardle, 1992). This formula was obtained on the assumption that a fish would keep its envelope of swimming motion amplitude constant at the trailing edge (Lighthill, 1960). Kinematic data on some swimming fishes show that a variable amplitude of the wave exists near the trailing edge, although accurate measurements of the movement at the tail end are rare and are complicated by the dorsoventral bending of the tail plate.

*To whom reprint requests should be addressed.

†Present address: Institut für Sportwissenschaft, Biomechanik, Friedrich Schiller Universität, Seidelstrasse 20, D-06649 Jena, Germany

If some species do not utilize a constant-amplitude envelope near the caudal end, it is worthwhile to determine to what extent this affects the value of propulsive quantities in the context of the elongated body theory. This point will be discussed in the present paper using kinematic data on saithe (Videler and Hess, 1984) and on two trout (Blickhan *et al.* 1992). We also compare these calculations with estimates obtained by applying the three-dimensional waving plate theory (Cheng *et al.* 1991).

Elongated body theory

It is assumed that the fish swims at speed U. In the fish-centred moving system of coordinates (x,y,z), the net flow is directed in the positive x-direction and the mean surface of the undulating fish is located in the x,y-plane. The z-axis points in the direction of the lateral undulation normal to the x,y-plane. The lateral movement of the body may be described by:

$$z = h(x,t), \tag{2}$$

where h is the lateral deflection and t is time.

If just the hydrodynamic force associated with the vortex sheet from the tail in the absence of any vortex shedding from the body fins (Wu, 1971; Yates, 1983) is considered then, according to the elongated body theory (denoted by EBT, Lighthill, 1960, 1975), the time-averaged values of thrust ($\bar{T}$), power required ($\bar{P}$) and efficiency (η) are:

$$\bar{T} = m(L)\left|\left(\overline{\frac{\partial h}{\partial t}w} - \frac{1}{2}\overline{w^2}\right)\right|_{x=L}, \tag{3}$$

$$\bar{P} = Um(L)\left|\left(\overline{\frac{\partial h}{\partial t}w}\right)\right|_{x=L} \tag{4}$$

and

$$\eta = \frac{U\bar{T}}{\bar{P}} = \frac{\left|\left(\overline{\frac{\partial h}{\partial t}w} - \frac{1}{2}\overline{w^2}\right)\right|_{x=L}}{\left|\left(\overline{\frac{\partial h}{\partial t}w}\right)\right|_{x=L}}, \tag{5}$$

where $m(x)$ represents the added mass per unit length, L is body length and

$$w = \frac{dh}{dt} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \tag{6}$$

is the relative fluid velocity at any cross section.

The frequently observed movement pattern consisting of a wave of lateral deflection running with increasing amplitude from nose to tail along the body of the fish may be approximately described by:

$$h(x,t) = h_1(x)\cos(kx - \omega t), \tag{7}$$

where $h_1(x)$ is the amplitude, $k=2\pi/\lambda$ is the wave number (λ being the wavelength) and $\omega=2\pi f$ is the circular frequency (f being the frequency).

Substituting equation 7 into equations 3–6, we have:

$$\bar{T} = \frac{1}{4} m(L)[\omega^2 h_1^2(L)\left|1 - \frac{U^2}{V^2}\right| - U^2(h_1')^2(L)],$$ (8)

$$\bar{P} = \frac{1}{2} U m(L)\omega^2 h_1^2(L)\left|1 - \frac{U}{V}\right|,$$ (9)

$$\eta = \frac{1}{2}\left|1 + \frac{U}{V}\right| - \frac{1}{2}\left[\frac{U}{\omega}\right]^2\left|1 - \frac{U}{V}\right|^{-1}\frac{[h_1'(L)]^2}{[h_1(L)]}$$ (10)

and

$$w = \left|1 - \frac{U}{V}\right|\frac{\partial h}{\partial t} + U\frac{h_1'(x)}{h_1(x)}h,$$ (11)

where $V = f\lambda = \omega/k$ represents the wave speed and $h_1' = \partial h/\partial x$ is the local slope.

Thrust and efficiency are obviously dependent on the value of $h_1'(L)$, i.e. the slope at the caudal end. This result was given in the first paper about the elongated body theory for fish swimming (Lighthill, 1960). In that paper, Lighthill suggested that to make η close to 1, it is desirable for $h_1'(L)$ to be practically zero because a non-zero value of $h_1'(L)$ reduces the thrust without altering the rate of working. Under this condition, the thrust, the efficiency and the relative fluid velocity become:

$$\bar{T}_0 = \frac{1}{4} m(L)\omega^2 h_1^2(L)\left|1 - \frac{U^2}{V^2}\right|,$$ (12)

$$\eta_0 = \frac{1}{2}\left|1 + \frac{U}{V}\right|$$ (13)

and

$$w_0 = \left|1 - \frac{U}{V}\right|\frac{\partial h}{\partial t}.$$ (14)

Later, Lighthill arrived at the above expressions by using equation 14 directly and considering the bulk momentum and energy change at the trailing edge of the fish (Lighthill, 1975; Blake, 1983).

Equations 12–14 are widely used by many investigators. However, this is based on the assumption that the animals should adjust $h_1'(L)$ to zero to achieve high efficiency without examination of whether this is, in fact, the case. The assumption may not be fulfilled in general. Furthermore, in EBT, all time mean quantities are determined by the fin shape and movement at the caudal end.

In the following section, we will present some examples where $h_1'(L)$ is not equal to zero. We will try to estimate the influence of the non-zero value of $h_1'(L)$ on propulsive efficiency as calculated from EBT. Furthermore, these values will be compared with the results predicted by a three-dimensional unsteady theory.

Influence of the slope at the tail end

Saithe

The first example is based on kinematic measurements from a swimming saithe (Videler and Hess, 1984), a subcarangiform swimmer. For this animal, the slope of $h_1(x)$ near the tail end is not equal to zero (Fig. 1). Hess and Videler (1984) noticed the influence of the amplitude slope at the tail end, but still considered a higher efficiency to be more plausible. However, the measurements were highly accurate and the contribution of the term containing the slope at the caudal end should be included in the estimation of thrust and efficiency.

The body dimensions of, and kinematic data on, an 'average' saithe needed for the calculation of hydrodynamic quantities are listed in Table 1. The wave speed V (or the wavelength λ) measured for the posterior half of the fish is used, as this part of the fish is largely responsible for propulsion. Equation 10 can be written as:

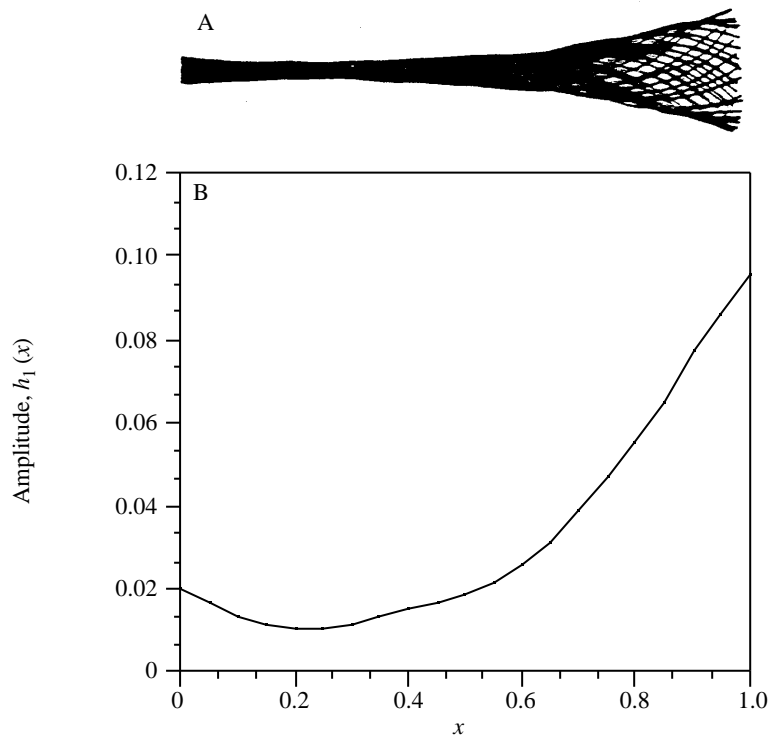$$\eta = \frac{1}{2}(1+\beta) - \frac{1}{2}\alpha^2 \frac{\beta^2}{1+\beta}, \tag{15}$$

Fig. 1. Amplitude envelope of the saithe taken from Videler and Hess (1984). (A) Digitized centre lines of a sequence of body movements (dorsal view) in swimming saithe, with the *x*-positions of the nose (at the left) made to coincide. (B) The lateral amplitude of movement for an 'average' saithe was obtained by averaging the measured quantities from all conducted sequences (Videler and Hess, 1984; Hess and Videler, 1984). One hundred equidistant points were used to calculate each centre line. Nose is at *x*=0, tail end at *x*=1 (unit: *L*).

where $\beta = U/V$ and

$$\alpha = \frac{\lambda}{2\pi} \frac{h_1'(L)}{h_1(L)}. \tag{16}$$

To determine $h_1'(L)$, we chose two points, $x=L$ and $x=L-\Delta L$, at which the amplitude values are h_m and h_t respectively. The value of $h_1'(L)$ is simply obtained from:

$$h_1'(L) = \frac{h_m - h_t}{\Delta L}, \tag{17}$$

giving:

$$\alpha = \frac{1}{2\pi} \left[\frac{\lambda}{L} \right] \left[\frac{\Delta L}{L} \right]^{-1} \left[1 - \frac{h_t}{h_m} \right]. \tag{18}$$

From Table 1, we take $\beta = 0.827$ and $\lambda = 1.04L$. If we chose $\Delta L = 0.04L$ then, from Fig. 1B, $h_t = 0.076L$ and $h_m = 0.083L$ can be estimated, giving $\alpha = 0.349$, $\eta_0 = 0.914$ and $\eta = 0.673$. In the case of the saithe, there is a considerable difference of up to 20% between the calculated values for efficiency obtained with and without a contribution from the slope.

Trout

The second and third examples will use our own kinematic data from two trout (trout 1, $L=0.16$m and trout 2, $L=0.18$m). Their kinematic data are listed in Table 1, in which the wavelength is obtained by directly measuring the wave form from the centre lines (Webb *et al.* 1984).

Table 1. *Variables for calculation of hydrodynamic quantities and the resulting efficiencies*

Variable	Abbreviation	Saithe	Trout 1	Trout 2
Body length (m)	L	0.40	0.16	0.18
Height of caudal fin at trailing edge (L)	b_t	0.24	0.20	0.20
Half-span (L)	b	0.12	0.10	0.10
Reynolds number	$Re = UL/\nu$	$2\times10^5 - 8\times10^5$	0.44×10^5	1.2×10^5
Time period (s)	T_t	0.278	0.30	0.21
Swimming speed (L/T)	U	0.86	0.516	0.777
Amplitude at the tail end (L)	h_m	0.083	0.100	0.104
Wave speed (caudal body) (L/T_t)	V	1.04	0.864	0.942
Speed ratio (caudal body)	$\beta = U/V$	0.827	0.598	0.817
Reduced frequency	$\sigma = 2\pi/U$	7.302	12.17	8.156
Wave number	$k = 2\pi/V$	6.038	7.269	6.667
Wavelength (caudal body) (L)	λ	1.04	0.864	0.942
Efficiency without slope term by using EBT	η_0	0.913	0.799	0.909
Efficiency with slope term by using EBT	η	0.673	0.738	0.724
Efficiency by using 3DWPT	η	0.602	0.686	0.573
Speed ratio specified by equation 19	β_m	0.670	0.652	0.697

EBT, elongated body theory; 3DWPT, three-dimensional waving plate theory.

The digitized centre lines of a sequence of movement and the amplitude function of trout 1 are shown in Fig. 2A,B. In one swimming period (T_p=0.38s), 71 centre lines are digitized. Forty segments are used to construct each line. The non-zero slope near the end of the tail can also be seen. The amplitude distribution shows an asymmetry with respect to the x-axis. The average values of the up and down (left and right movements in reality) amplitudes are plotted in Fig. 2B and will be used in the calculation of locomotory dynamics.

From Table 1, we have β=0.598 and λ=0.864L. If we chose ΔL=0.10L, then h_t=0.073L and h_m=0.100L. So α=0.371, η_0=0.799 and η=0.738. In this example, the influence of the actual slope on the calculated efficiency is less pronounced.

The movement of trout 2 is shown in Fig. 3. Again the average amplitude of up and down movements is used (Fig. 3B). From the table and figure, we have β=0.817, λ=0.942L, ΔL=0.05L h_t=0.092 and h_m=0.104L. Hence, α=0.318, η_0=0.909 and η=0.724. The efficiency changes considerably after considering the slope term for trout 2.

Comparison with the three-dimensional unsteady theory and discussion

The three-dimensional waving plate theory (denoted by 3DWPT, Cheng *et al.* 1991) can be used to obtain a more accurate estimate of the propulsive performance of the examples given above. In this theory, the incompressible potential flow past a flexible
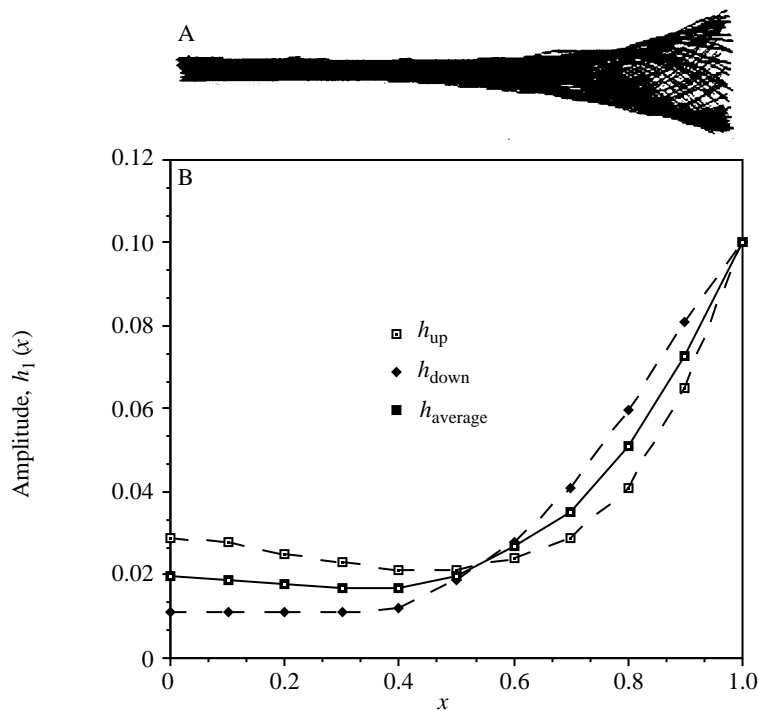

Fig. 2. Amplitude envelope of trout 1. (A) Digitized centre lines of a sequence of body movements for trout 1 swimming on the spot in a water tunnel. (B) Lateral amplitude of the movement for trout 1. Forty equidistant points were used to calculate each centre line.

thin plate of finite aspect ratio performing a small-amplitude undulatory motion is treated by the linear unsteady vortex ring panel method in the frequency domain. The plate and its wake are replaced by a suitable distribution of vortex rings in a way satisfying the appropriate boundary conditions.

In the calculations we present, the swimming fishes are modelled by rectangular undulating plates (length L and height $2b$) corresponding to the centre surfaces of their bodies. The body movement is assumed to be described by equation 7 with amplitude envelopes $h_1(x)$ (Figs 1B, 2B and 3B) that are close to exponential functions (Cheng and Blickhan, 1994). The given parameters are reduced frequency $\sigma = \omega L/U$ and wave number $k = 2\pi L/\lambda$ and the calculated efficiencies are listed in Table 1. It is found that, for each fish, the value of efficiency calculated by 3DWPT is lowest among three results. The efficiency of EBT with the slope term is close to that of 3DWPT, i.e. the slope term should be considered in using EBT for these fishes.

In EBT, the time-averaged propulsive quantities depend only on the events at the caudal end. These estimates are based on the assumption that movement and body shape immediately rostral to the tail end should not be very different ('slender body'). As shown, the change of movement near the tail tip cannot be neglected for fishes swimming with variable amplitude. The EBT contains a term that approximately considers the influence of the slope. In practice, when determining the slope of the amplitude envelope at the caudal end, a segment as large as the whole tail might be considered.

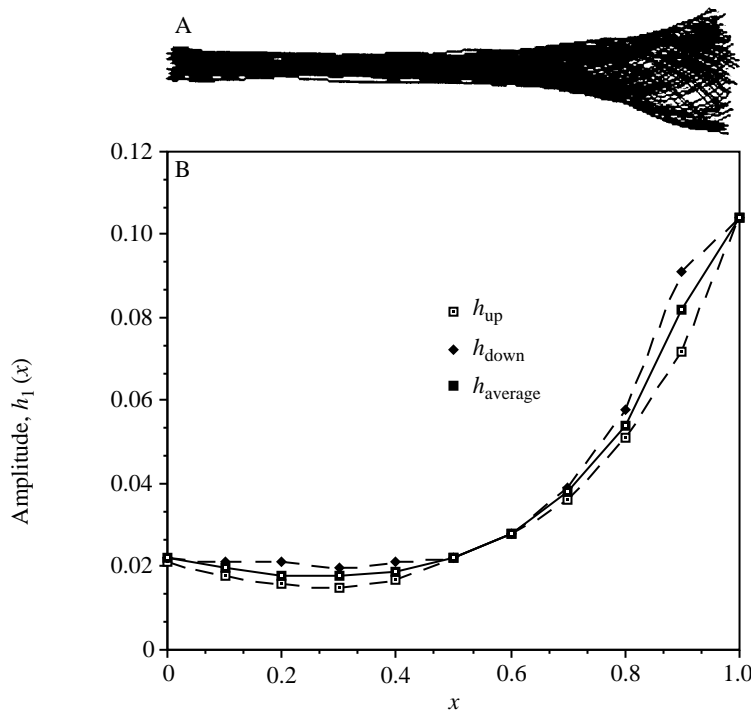What is the true efficiency achieved by swimming fishes? The opinion that many

Fig. 3. Amplitude envelope of trout 2. See Fig. 2 for details.

species swim at efficiencies as high as 90% is based on hydrodynamic calculations. The EBT predicts that a zero slope near the body end is desirable, but this has not been achieved by many fishes, especially those swimming in the subcarangiform mode. For those animals, efficiency should be significantly reduced. Perhaps fishes and cetaceans swimming in the carangiform mode use such a movement pattern and achieve efficiencies of around 90% with lunate fins or flukes.

The increase of the deflection amplitude at the caudal end has an important consequence. If the slope at the tail end is not equal to zero, then efficiency is not simply proportional to β, the ratio of swimming speed to wave speed (Cheng *et al.* 1991). From equation 15, we can find for a given relative slope α a value of the speed ratio β_m at which the efficiency reaches its maximum (see Table 1). Let $\partial\eta/\partial\beta=0$ and considering $\beta=U/V<1$, we have:

$$\beta_m = 1 - \left[1 - \frac{1}{1+\alpha^2}\right]^{1/2}. \tag{19}$$

We have used the data from the saithe to see how the efficiency changes with the ratio β when the slope term is included. The wavelength of the waving movement of the body of the fish is almost unchanged when swimming at different speeds for the same fish (Bainbridge, 1963; Webb *et al.* 1984). Thus, the slope near the tail end of saithe is largely unchanged and the value of α remains constant. The EBT, with the slope term included in the analysis, and 3DWPT give similar results at high values of β, and both depart significantly from the results of EBT without the slope term (Fig. 4). For movement

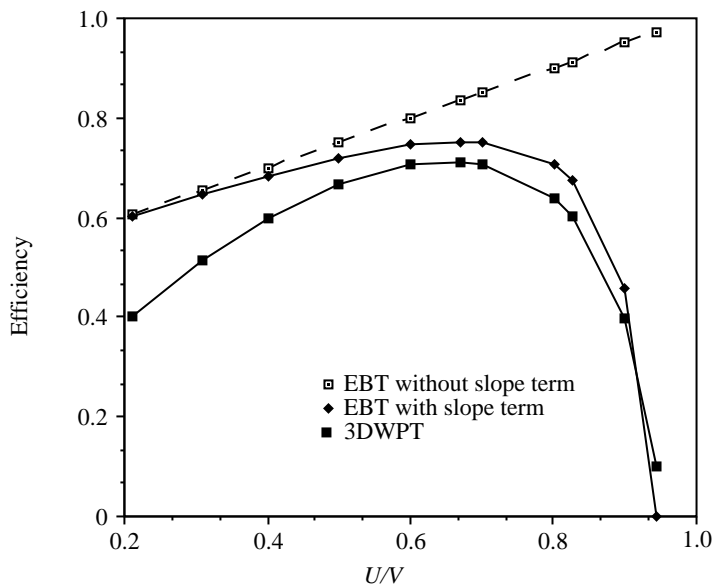Fig. 4. The variation of efficiency with the ratio of swimming speed to wave speed (β) at α=0.349 calculated by the elongated body theory (EBT), without and with the slope term, and by the three-dimensional waving plate theory (3DWPT).

patterns with variable amplitude, the EBT with the slope term included in the calculations can be considered as suitable for the estimation of swimming performance.

Obviously, both EBT and 3DWPT are limited by their basic assumptions. In order to obtain more precise estimates of hydrodynamic quantities, more realistic fluid-dynamic models are needed.

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