MECHANICAL WORK RATE MINIMIZATION AND FREELY CHOSEN STRIDE FREQUENCY OF HUMAN WALKING: A MATHEMATICAL MODEL

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Summary

The interplay between the work done to move the body centre of mass with respect to the environment (external work) and the work done to move the limbs with respect to the body (internal work) has been shown experimentally partially to determine the freely chosen stride frequency during walking. A mathematical model that estimates the two components of the mechanical work is proposed. The model, according to the criterion of work rate minimization (both positive and positive plus negative), is able to predict the natural stride frequency as a function of the average progression speed. The adequacy of the model and the validity of the assumptions have been checked against measurements of natural stride frequency in 11 subjects walking on a treadmill at several speeds (range $1-3\,\mathrm{m\,s^{-1}}$). Comparison with theoretical predictions shows good agreement with the minimization of positive work rate at low speeds, while at high speeds the stride frequency is better explained by the model for minimum positive plus negative work rate.

Introduction

Optimization principles seem to control many biological functions, particularly repetitive processes. The hypothesized 'controller' has to choose from an appropriate number of extensive variables to satisfy some optimization criteria: e.g. performance maximization, energy saving, comfort or safety maintenance (Hämäläinen, 1978). In this respect, simple mathematical models can be used in parallel with experimental work to design or verify hypotheses about the optimization mechanisms.

In the past such modelling has been applied in studies of the respiratory (Otis *et al.* 1950; Mead, 1960; Hämäläinen and Viljanen, 1978) and cardiovascular systems (Yamashiro *et al.* 1979; Hämäläinen and Hämäläinen, 1985), whereas investigators of human locomotion (Margaria, 1938; Cavagna *et al.* 1977; Cavagna and Franzetti, 1986; Alexander, 1989) have shown experimentally that optimization phenomena also occur in gait. There is a striking similarity between breathing

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and walking, as pointed out by Cavagna and Franzetti (1986). In order to cope with particular performance requirements (alveolar ventilation and progression speed) the system has to select the appropriate combination of extensive parameters (tidal volume/breathing frequency, step length/step frequency) following an optimization criterion, for instance the minimization of mechanical work rate or average force. In both systems, for a fixed ventilation rate or speed, the total work curves attain high values at the extremities of the frequency range, whereas somewhere in the middle they have a point which corresponds to the minimum work rate. In breathing, such behaviour is produced by a balance between the viscous and elastic components of the total work done, each of which increases or decreases monotonically with an increase in breathing frequency. During walking, *mutatis mutandis*, the total work components displaying the same trend are the internal work rate ($\dot{W}_{int}$), due to the movements of the limbs with respect to the body centre of mass, and the external work rate ($\dot{W}_{ext}$), due to the displacement of body centre of mass with respect to the environment. The rationale underlying the choice of a certain stride frequency might be the following: when walking at a constant speed using low-frequency longer steps, most of the energy expended is used to raise and lower the body centre of mass, resulting in a greater external component, while at the highest stepping frequencies the body centre of mass moves along a relatively straight line and most of the energy is devoted to accelerating and decelerating limbs with respect to it, resulting in a greater internal component. Cavagna and Franzetti (1986) showed experimentally that the total mechanical work rate ($\dot{W}_{tot}$), obtained by adding $\dot{W}_{int}$ and $\dot{W}_{ext}$, at each constant progression speed displays a minimum for a step frequency roughly corresponding (-25%) to the one freely chosen by the subjects. Their predictions have been further improved (-12%) by using more recent (Dempster *et al.* 1959) average anthropometric data (Minetti *et al.* 1990a).

Despite the multiplicity of methods available for calculating the mechanical work of locomotion (which yield results varying over three orders of magnitude; Williams and Cavanagh, 1983), the technique adopted by Cavagna and Franzetti (1986) seems to be the only one to face the optimization phenomena related to the chosen stride frequency. There is thus a need to incorporate such a method into a comprehensive and simple mathematical model capable of verifying the assumptions about minimization of the mechanical work rate during walking.

The aim of the present study is to provide simple equations for $\dot{W}_{int}$ and $\dot{W}_{ext}$ and to compare the model predictions of natural stride frequency with experimental data.

Materials and methods

Fig. 1 shows the stick diagram adopted in the formalization of the $\dot{W}_{int}$ model. The body is composed of five stiff segments, four of which (two lower and two upper limbs) are involved in the model computation. The values of segment
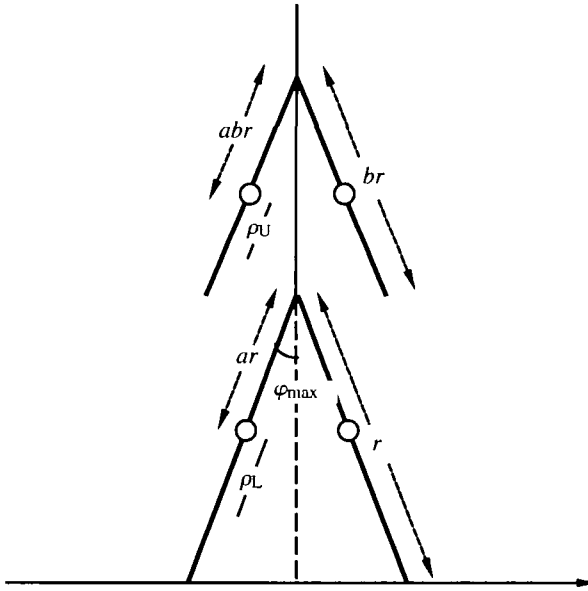
Fig. 1. Stiff limb diagram adopted in the model formalization, where *r* is lower limb length, *a* is the proximal distance of the lower limb centre of mass (as a fraction of *r*), *b* is upper limb length (as a fraction of *r*), and ρ_L and ρ_U are the radii of gyration of upper and lower limbs. The circles represent the approximate position of the limb centre of mass. All the equations have been modelled assuming a sinusoidal function over time for the angle between the limbs and the vertical line (total excursion$=\pm\varphi_{max}$).

length, mass, position of the centre of mass (circles in Fig. 1) and radius of gyration have been taken from the literature (Dempster *et al.* 1959).

Assuming that the limb extremities follow a sinusoidal displacement with respect to the head–trunk segment during walking, $\dot{W}_{int}$ has been evaluated from the oscillations in kinetic energy (both translational and rotational) according to the König theorem, as suggested by Cavagna and Kaneko (1977). In fact, the energy transfer among segments is not relevant because of the in-phase shapes of the kinetic energy curves. The derived equation (see Appendix) is:

$$\dot{W}_{int} = f\bar{s}^2\frac{\pi^2}{2}[(a^2 + g_0^2)(m_L + b^2 m_U)], \tag{1}$$

where *f* is stride frequency, $\bar{s}$ is average progression speed, *a* is the proximal distance of the lower limb centre of mass, g_0 is mean radius of gyration, m_L and m_U are the masses of the lower and upper limbs and *b* is the upper limb length. For a 'standard' subject, this reduces to:

$$\dot{W}_{int} = 21.638f\bar{s}^2. \tag{2}$$

Unlike this $\dot{W}_{int}$ calculation, the $\dot{W}_{ext}$ estimate cannot be modelled using the
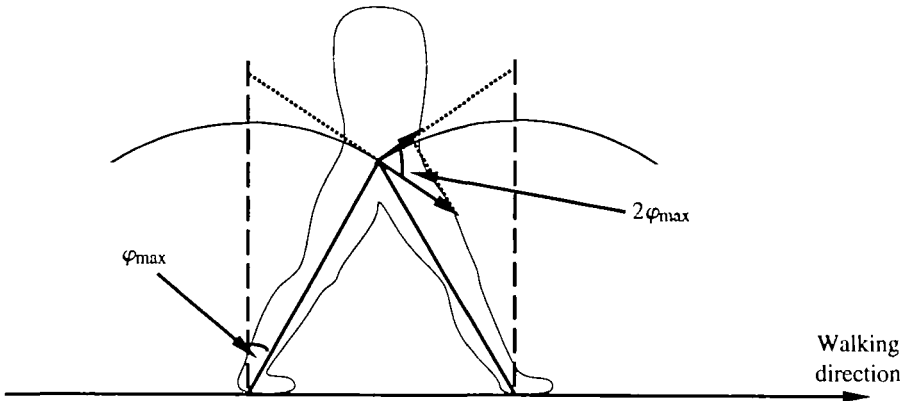
Fig. 2. Geometry of the trajectories of the body centre of mass in the transition between successive steps. The energy at the end of one step is degraded proportionally to the sine of the angle between the speed trajectory and the orientation of the forward lower limb. φ_{max} is the maximum angle between the vertical and the limbs.

technique given by Cavagna and Kaneko (1977). Modelling the movement of the body centre of mass as an inverted pendulum (Alexander, 1976), precludes calculation of $\dot{W}_{ext}$ based on the oscillations of its potential and kinetic energies, for they sum to a value invariant with respect to time (no energy range is displayed). Nor can we consider simply the changes in kinetic energy due to forward speed changes of the inverted pendulum because, if its movement were modelled so as to mimic that of the lower extremity, it would result in a speed range from zero to a certain value, a behaviour not encountered in normal walking. Such approximation would lead to substantial overestimations of $\dot{W}_{ext}$.

In this paper the calculation of $\dot{W}_{ext}$ is based on the following rationale: the only fluctuations in the energy curve of the body centre of mass (concentrated in the hip joint) to be taken into account are those produced by the loss of energy occurring between the end of a step and the beginning of the next one due to the angle formed by the trajectory vectors in that transition. When the maximum angle between the front and the hind limbs ($2\varphi_{max}$ in Fig. 2, where φ_{max} is the maximum angle between the vertical and the limbs) is zero the body centre of mass energy could ideally be completely recovered, but when it is $\pi/2$ the trajectory vector will be in line with the front limb, precluding any energy transfer to the subsequent step. On the basis of an elastic collision, the energy loss in the forward direction is proportional to $\cos^2(2\varphi_{max})$, but we opted, for a number of reasons (see Results and Discussion), to approximate the function with a simple cosine.

When the external work is modelled taking the above considerations into account (see Appendix), we reach the following estimate:

$$\dot{W}_{ext} = \frac{m\bar{s}^4}{8fr^2},\tag{3}$$

where m is body mass and r is lower limb length. For a 'standard' man, this reduces to:

$$\dot{W}_{ext} = 10.171 \frac{\bar{s}^4}{f}.$$ (4)

Summing the internal and external components, we obtain an expression for the total mechanical work rate:

$$\dot{W}_{tot} = \dot{W}_{int} + \dot{W}_{ext} = \bar{s}^2 \left\{ \frac{f\pi^2}{2}[(a^2 + g_0^2)(m_L + b^2 m_U)] + \frac{m\bar{s}^2}{8fr^2} \right\},$$ (5)

which, properly manipulated (see Appendix), allows calculation for different values of $\bar{s}$ the stride frequencies (f_{opt}) at which the work rate is minimized:

$$f_{opt} = \frac{\bar{s}}{2\pi r} \sqrt{\frac{m}{(a^2 + g_0^2)(m_L + b^2 m_U)}}.$$ (6)

Measurements of the natural stride frequency were carried out on 11 healthy subjects (mass 48–92 kg, height 1.63–1.81 m, age 19–62 years) walking on a treadmill at different speeds (range 1–3 m s^{-1}).

Results and discussion

The adequacy of the model has been tested by comparing our graphs of $\dot{W}_{int}$, $\dot{W}_{ext}$ and $\dot{W}_{tot}$ *versus* stride frequency at constant progression speed (Fig. 3) with the ones provided by Cavagna and Franzetti (1986; Fig. 2B) at the same speeds (1.278, 1.5 and 1.8 m s^{-1}). The most crucial difference in the work rate curves occurs for $\dot{W}_{int}$, which is smaller and linear in the present study, while the experimental data show a curved relationship with stride frequency. This discrepancy is due to the different methodology adopted in the estimation of the internal work rate. Cavagna and Franzetti (1986) took the displacement curves of four subjects investigated in a previous experiment (Cavagna and Kaneko, 1977), who walked at speeds ranging from about 1 to 5.5 m s^{-1} at their natural stride frequency, and forced those curves, by re-scaling the time axis, to correspond to three given speeds [4.6, 5.5 and 6.5 km h^{-1} using their units (1.3, 1.5 and 1.8 m s^{-1}, respectively)], maintaining the original step length range at each new speed. In so doing, they clustered estimates of $\dot{W}_{int}$ calculated from very different motion patterns of the limbs at every constant speed.

In our model only the stride frequency was allowed to change, while the motion pattern was invariant (a sinusoidal oscillation). This treatment implies, for a given progression speed, a constant maximum speed of the limbs with respect to the body centre of mass, regardless of the stride frequency adopted, causing $\dot{W}_{int}$ to be a linear function of f (within the limits indicated by equation A18). Another reason for the observed discrepancy in $\dot{W}_{int}$ is the fact that Cavagna and Franzetti (1986) adopted in their computations the anthropometric data of Braune and Fischer (1892), which report the mass of four limbs equal to 50.22 % of the entire body
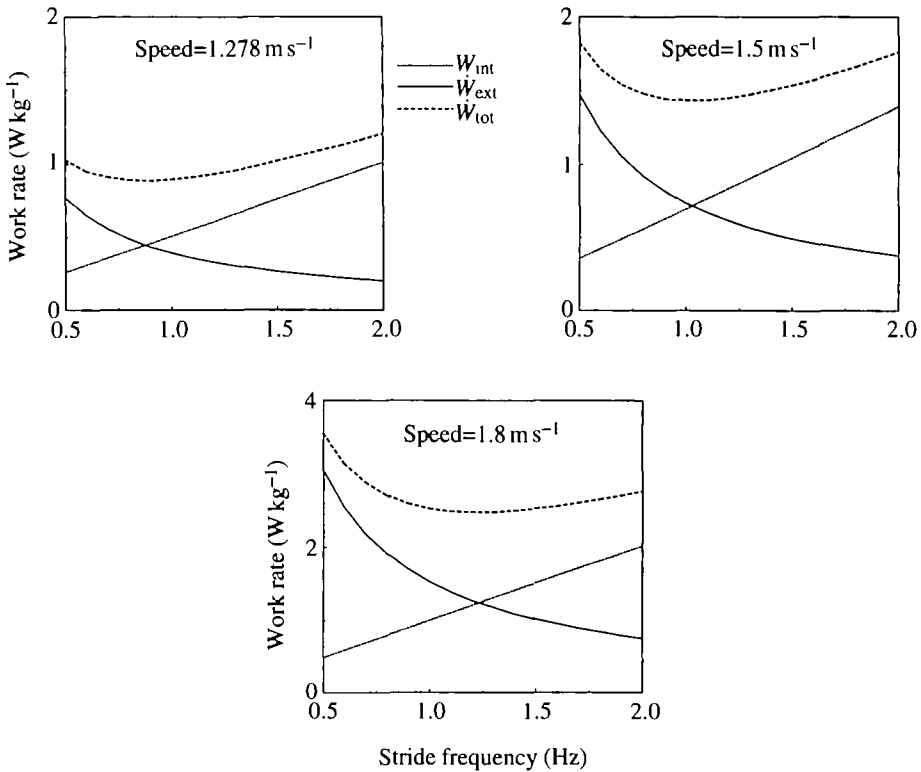
Fig. 3. Positive internal ($\dot{W}_{int}$), external ($\dot{W}_{ext}$) and total ($\dot{W}_{tot}$) work rates at three selected speeds plotted against stride frequency calculated using the present model. The speed values have been chosen to allow comparison with the experimental data provided by Cavagna and Franzetti (1986).

mass, while more recent measurements (Dempster *et al.* 1959) set this at 42.20 %. This difference has been shown to be responsible for a 20 % overestimation of the internal work rate (Minetti *et al.* 1990a). Recently, Minetti and Saibene (1992) found good agreement between the preliminary predictions of the $\dot{W}_{int}$ model and experimental data for freely chosen and forced stride frequencies at different walking speeds (range 0.69 to 2.08 m s^{-1}).

Conversely, modelling the external work according to $\cos^2(2\varphi_{max})$ (the energy loss due to the impact during the step transition) leads to unrealistic high work values when compared with the experimental results of Cavagna and Franzetti (1986). We adopted a simple $\cos(2\varphi_{max})$ function, which behaves similarly but attenuates the energy losses, because it is supported by three circumstances occurring during walking. First, at low step frequencies (and longer step lengths) the horizontal trunk rotation separates the hips, corresponding to an increase in the lower limb length. This increase in r results in a decrease in the external work rate while having no effect on the internal one, whose equation is unaffected by changes in the lower limb length. Second, the spread of the four limbs moves the

body centre of mass vertically with respect to the hip, reducing the angle between the lines of impact. When modelling this aspect with the walking geometry assumed in this study and standard anthropometric data, it turns out that the vertical range of the movement of the body centre of mass is about 60 % of the hip movement range. Finally, foot extension acts to smooth the step transition, by increasing the radius before the collision. All three occurrences imply a decrease in $2\varphi_{max}$ and, concurrently, in the energy losses related to the step transition, resulting in a lower external work rate.

The $\dot{W}_{ext}$ values obtained using this approach, whose rationale is based on observations about the energy loss of walking made by others (Cavagna *et al.* 1977), nevertheless resemble both quantitatively and qualitatively those obtained by Cavagna and Franzetti (1986), who used a force platform. Only at high speed is there a tendency to overestimate the experimental data because of the fourth power in equation A24.

The mathematical model proposed in the present paper predicts the relationship between average walking speed and the natural stride frequency according to the criterion of minimum work rate (Fig. 4) (curve $\dot{W}_{min}$, 'standard' subject). These predictions seem to be in agreement with our experimental data at low speeds, but deviate considerably when the speed increases. Although Strathy *et al.* (1983) found that treadmill walking overestimates the stride frequency by about 7 %, the accuracy of our measurements is confirmed by values reported by others (Cavagna and Franzetti, 1986). It is important to keep in mind that the $\dot{W}_{min}$ curve reports the stride frequencies at which the rate of positive internal work is minimal during the stride (only the positive differences in kinetic energy are included in the $\dot{W}_{int}$ computation) and does not take into account the work necessary for decelerating the limbs. From another viewpoint, the same curve can be regarded as the one that minimizes the positive and negative internal work rates pertaining only to one stride phase, for example the stance, while the other phase (the swing) is considered as a ballistic (i.e. passive) movement, a hypothesis also suggested by others (Mochon and McMahon, 1980). In Fig. 4 a horizontal line marks the frequency at which a frictionless pendulum (with length equal to the mean of the lower and upper limb proximal distance of the centre of mass) oscillates naturally. Above this frequency the limbs have to be continuously accelerated and decelerated, with increasing positive and negative work (see below).

The calculation of $\dot{W}_{tot}$ partly follows the algorithm proposed by Cavagna and Kaneko (1977) but, as anticipated in the Introduction, there does not seem to be agreement about the computational technique to adopt when estimating the mechanical work of locomotion (Williams and Cavanagh, 1983; Minetti *et al.* 1990*b*). The major uncertainties reside in the energy transfer among segments, the incorporation of negative work and the recovery of the stored elastic energy. For this reason we incorporate in Fig. 4 a shaded area whose right-hand boundary, with equation $f_{opt}=0.485\bar{s}$, represents the minimization of the total work rate whose internal component has been doubled, thus also taking into account the negative work rate of both stride phases. Such high weighting for the negative
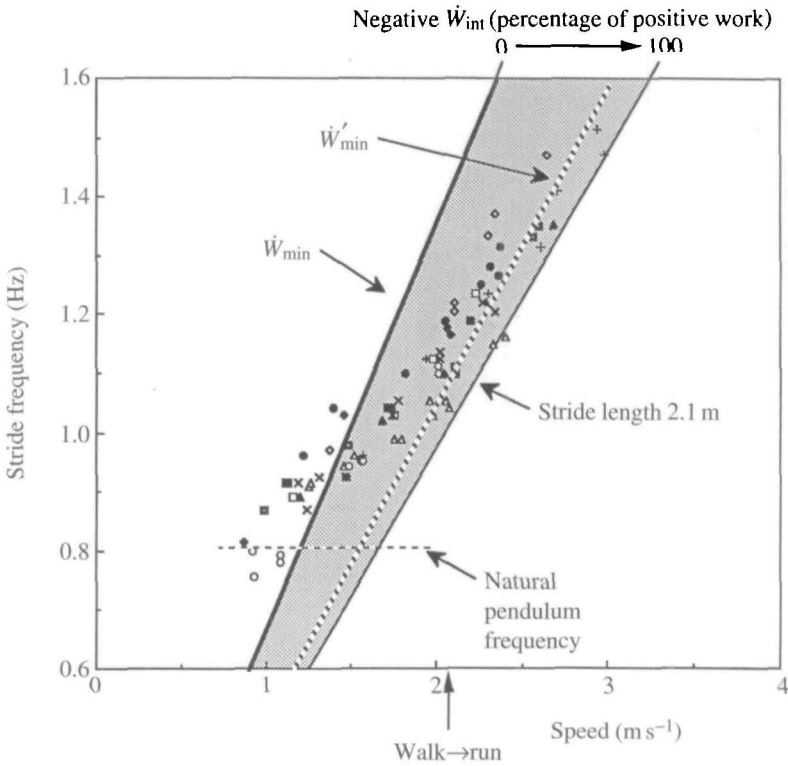
Fig. 4. Experimental data from 11 subjects (one symbol type per subject) together with model predictions. Lines labelled $\dot{W}_{min}$ and $\dot{W}'_{min}$ represent the frequencies minimizing the mechanical work rate when r is equal to the leg and the leg plus foot length, respectively. The stippled area shows the effect of different negative work efficiency in moving the predicted line to the right (see the text for further explanations).

work (100 % of positive $\dot{W}_{int}$) has been used in another study on the estimation of the mechanical work of locomotion (Winter, 1979), although other investigators (Williams and Cavanagh, 1983) adopted a value of about 33 % (that is to say, they considered the efficiency of negative work as being three times greater than the efficiency of positive work, provided that, during level walking, positive work equalled negative work). The shaded area within the two lines includes the curves which minimize total mechanical work with different negative work efficiencies. Within this area, a departure from $\dot{W}_{min}$ towards the right-hand limit could be interpreted as the work minimization (positive plus negative) when one stride phase (the swing) becomes progressively less passive. The last assumption about two active phases (no ballistic return) is mirrored by the behaviour of the respiratory system during increased alveolar ventilation, when expiration can no longer be considered to be a passive return to the initial operating condition (Otis et al. 1950).

All frequency predictions made by the present model are linear functions of the progression speed, resulting in a constancy of stride length (step length 0.73 and 1.03 m for $\dot{W}_{min}$ and the right-hand limit, respectively). These findings are not reflected by the experimental points which, although almost linear, seem to intersect several iso-length lines when the speed increases. As stated above, those lines pertain to different weightings of negative internal work (or different proportions in the activity of the swing phase). However, when a subject is asked to walk at speeds normally achieved by running ($>2.1\,\mathrm{m\,s^{-1}}$), a tendency to maintain a constant step length is found.

Another phenomenon only partially taken into account by the model is the foot extension between successive steps that occurs at high walking speeds. This results both in a reduction of energy loss due to the angle between the trajectory vectors (see Fig. 2), as suggested by Cavagna *et al.* (1977) (Fig. 3), and in an increase in the system energy. To simulate the first effect we included a greater segment length (lower limb plus foot) in the $\dot{W}_{ext}$ computation and obtained the prediction labelled $\dot{W}'_{min}$ in Fig. 4 ($f_{opt}=0.533\bar{s}$), which corresponds to a lower optimal stride frequency.

A stability analysis of our model predictions reveals a marked insensitivity to changes in body mass (only a different mass distribution within the body plays a role in the $\dot{W}_{int}/\dot{W}_{ext}$ balance). However, an increase in subject height results in lower optimal stride frequencies. Fig. 5 shows the $\dot{W}_{min}$ and $\dot{W}'_{min}$ curves given in
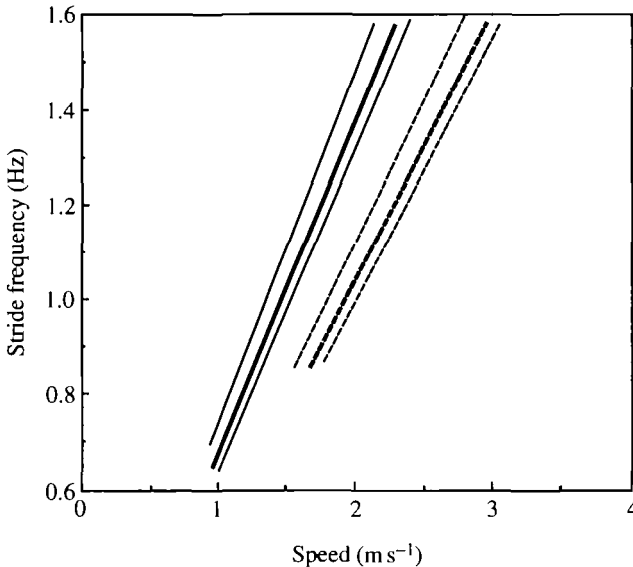

Fig. 5. Limits for the shortest and the tallest subjects of our group (left-hand and right-hand thin lines, respectively) related to the predicted natural stride frequency based on the criterion of the minimum work rate. Solid line, segment length=lower limb; dashed line, segment length=lower limb plus foot.

Fig. 4 (thick solid and dashed lines) together with the limits, *ceteris paribus*, for the tallest and the shortest subjects of our group (right-hand and left-hand thin lines, respectively). The narrow bands between the corresponding thin lines reflect the variability of the experimental data in Fig. 4, which do not show height-dependence in the scatterplot.

The reasons for the discrepancies lie partly in the assumptions within the model (stiff limbs, no double support, body centre of mass located in the hip joint) and partly in the possibility that the minimum work rate criterion is not the only one involved in performance optimization.

A recent paper about the energy expenditure of locomotion in mammals argues that a force minimization criterion is probably operating (Kram and Taylor, 1990) in species of very different sizes. Such a hypothesis, supported by measurements of foot–substratum contact time during running, could not apply to walking because of the different gait mechanism (Cavagna *et al.* 1977). As pointed out by Alexander (1991), who reviewed several optimization criteria that could operate during locomotion, future models will have to consider both energetic aspects: the energy spent to generate work and the energy necessary to generate (isometric) force. In addition, our model of $\dot{W}_{ext}$ is very similar to the one proposed by Alexander (1991), which applies the principle of conservation of angular momentum, as also suggested by McGeer (1990). We tried to incorporate Alexander's equation in our model and found that it overestimates the experimental data of Cavagna and Franzetti (1986). However, because of the additive procedure for total work rate computation, the incorporation of that model results in comparable predictions about the optimal stride frequency.

The present model could be further improved by taking into account a frictional component and allowing for a penalty factor related to the departure of the stride frequency from the one for the ideal pendulum at increasing speeds. In addition, accurate measurements could quantify the energy-saving effects of trunk rotation, foot extension and segmental mass displacements as contributors to the elevation of the centre of rotation, with a consequent decrease in $2\varphi_{max}$ (see Fig. 2). By increasing the values and curvature of $\dot{W}_{int}$ and decreasing $\dot{W}_{ext}$, these considerations are expected to improve the model predictions about the optimal stride frequency.

Appendix

Equation for $\dot{W}_{int}$

Following the stick diagram reported in Fig. 1, the average progression speed, $\bar{s}$, can be formalized as:

$$\bar{s} = f4r\sin\varphi_{max}, \tag{A1}$$

where f is the stride frequency, r is the leg length, and φ_{max} is half the maximum inner angle between the lower limbs. Assuming that the limb extremities follow a sinusoidal displacement with respect to the body centre of mass, which is placed in the head–trunk segment and does not move horizontally because of the symmetri-

cal position of the limbs, the function describing the angle between the vertical and a limb during the stride is:

$$\varphi(t) = \varphi_{max}\sin(\omega t),\tag{A2}$$

where

$$\omega = 2\pi f,\tag{A3}$$

and t is time. Calculation of $\dot{W}_{int}$ needs both the translational and the rotational components of the total kinetic energy function. The lower limb extremity moves according to:

$$s_L(t) = r\sin(\varphi_{max}\omega t).\tag{A4}$$

The linear approximation of equation A2 is justified by the fact that we are interested in the maximum speed reached, which occurs at an angle close to $0°$ (with the leg in the vertical position).

Multiplying equation A4 by a, the proximal distance of the lower limb centre of mass (expressed as fraction of limb length), and differentiating with respect to time, yields:

$$\dot{s}_{Lcom}(t) = \frac{d[s_{Lcom}(t)]}{dt} = ar\varphi_{max}\omega\cos(\varphi_{max}\omega t),\tag{A5}$$

where $\dot{s}_{Lcom}$ is the movement of the centre of mass of the lower limb.

The equation for the translational kinetic energy for the two lower limbs (we can multiply by 2 because, although the speeds of the two segments are opposite in sign, the square values are in phase), is then:

$$TKE_{Lcom}(t) = 2\tfrac{1}{2}m_L\dot{s}_{Lcom}^2 = m_L(ar\varphi_{max}\omega)^2\cos^2(\varphi_{max}\omega t),\tag{A6}$$

where m_L is mass of the lower limb. Equation A6 shows a minimum equal to zero when the limbs reverse their movement, i.e. when $\varphi = \pm\varphi_{max}$, while the maximum is reached when $\varphi_{max}\omega t$ is zero, i.e. when the limbs are vertically aligned. In that position, because of the assumptions about the displacement of the extremities, only the horizontal kinetic energy has to be considered, since the vertical speed of the centre of mass of the limbs is zero.

Hence, the range in the translational kinetic energy is

$$\Delta TKE_{Lcom} = m_L(ar\varphi_{max}\omega)^2.\tag{A7}$$

Following this argument for the upper limb yields:

$$s_{Ucom}(t) = -abr\sin(\varphi_{max}\omega t),\tag{A8}$$

where b is the upper limb length as a fraction of the lower limb length. The translational kinetic energy range is:

$$\Delta TKE_{Ucom} = m_U(abr\varphi_{max}\omega)^2,\tag{A9}$$

while that related to four limbs, obtained by summation (by virtue of the in-phase shapes of equations A7 and A9) is:

$$\Delta TKE_t = \Delta TKE_{Lcom} + \Delta TKE_{Ucom}$$
$$= a^2r^2\varphi_{max}^2\omega^2(m_L + b^2m_U).\tag{A10}$$

Calculation of the rotational kinetic energy associated with the movement of the limbs requires a function describing the changes in angular speed within the stride duration. By differentiating equation A2 with respect to time:

$$\dot{\varphi}(t) = \frac{d[\varphi(t)]}{dt} = \varphi_{max}\,\omega\cos(\omega t). \tag{A11}$$

Calculating the moments of inertia for the lower and upper limbs (I_L and I_U) as

$$I_L = m_L \rho_L^2,$$

where

$$\rho_L = g_L r,$$

and

$$I_U = m_U \rho_U^2,$$

where

$$\rho_U = g_U b r, \tag{A12}$$

where ρ_L and ρ_U are the lower and upper limb radii of gyration around the centre of mass and g_L and g_U are the lower and upper limb radii of gyration expressed as fractions of segment length, we obtain the rotational kinetic energies:

$$RKE_L = 2\tfrac{1}{2}I_L\,\dot{\varphi}(t)^2 = m_L(g_L r)^2[\varphi_{max}\,\omega\cos(\omega t)]^2 \tag{A13}$$

and

$$RKE_U = 2\tfrac{1}{2}I_U\,\dot{\varphi}(t)^2 = m_U(g_U b r)^2[\varphi_{max}\,\omega\cos(\omega t)]^2. \tag{A14}$$

As for the computation of translational kinetic energy, we are interested in the range of rotational kinetic energy during the stride, which, from equations A13 and A14 sums to:

$$\Delta RKE_t = r^2\varphi_{max}^2\,\omega^2(m_L g_L^2 + m_U b^2 g_U^2). \tag{A15}$$

Thus, the equation for the rate of mechanical internal work, made up of the components of equations A10 and A15, is:

$$\dot{W}_{int} = 2f(\Delta TKE_t + \Delta RKE_t)$$
$$= 2fr^2\varphi_{max}^2\,\omega^2[a^2(m_L + b^2 m_U) + (m_L g_L^2 + m_U b^2 g_U^2)]. \tag{A16}$$

The ranges are multiplied by 2 because during the stride there are two minima and two maxima. Thus, only the positive work is taken into account (see Discussion).

By assuming the approximation:

$$\varphi_{max} = \arcsin\left(\frac{\bar{s}}{4fr}\right)$$

$$\approx \frac{\bar{s}}{4fr}, \tag{A17}$$

allowed because:

$$\frac{\bar{s}}{4fr} < \frac{\pi}{4} \rightarrow \frac{\bar{s}}{2f} = \text{step length} < \frac{\pi}{2}r \approx 1.5r, \tag{A18}$$

(a fairly unrestrictive condition), $\dot{W}_{int}$ can be expressed in terms of f and $\bar{s}$ as:

$$\dot{W}_{int} = f\bar{s}^2\,\frac{\pi^2}{2}\,[a^2(m_L + b^2 m_U) + (m_L g_L^2 + m_U b^2 g_U^2)]. \tag{A19}$$

A further simplification, allowed when lower and upper limbs show similar radii of gyration (as in the human body), is to average g_L and g_U into a single value, g_0, yielding:

$$\dot{W}_{int} = f\bar{s}^2 \frac{\pi^2}{2}[(a^2 + g_0^2)(m_L + b^2 m_U)].$$ (A20)

Expressing f in Hz and $\bar{s}$ in m s^{-1} and allowing for standard anthropometric values, equation A20 gives the following estimate of $\dot{W}_{int}$ in watts:

$$\dot{W}_{int} = 21.638 f\bar{s}^2.$$ (A21)

Equation for $\dot{W}_{ext}$

The total energy associated with the body centre of mass (TE$_{max}$) is considered to be constant during the oscillation of the lower limb (a single step). Thus, as for an inverted pendulum, TE$_{max}$ is the maximum energy level reached by the body centre of mass (assumed to be concentrated in the hip joint). The difference between the minimum energy level (TE$_{min}$) and TE$_{max}$ generates the external work; in this study it is assumed to be determined by the transition between successive steps (see Fig. 2), regarded as a type of elastic collision. Complete energy recovery, in the forward direction, can theoretically be feasible whenever the angle between the trajectory vector at the end of a step and at the beginning of the next one is near to 0° (i.e. when the body centre of mass moves along a relatively straight path). Conversely, the energy recovery is zero when the trajectory of the previous step is in line with the front limb segment which, owing to the fixed constraint with the ground, will need extra energy to move in the forward direction. Between these two extremes, the speed vector in the forward direction at the beginning of the next step is a fraction of the previous one according to a sine function of $(\pi/2 - 2\varphi_{max})$, i.e. of the angle between the trajectory vector and the orientation of the front limb at the step transition (see the vector projection onto the path of the next step in Fig. 2). While on the basis of an elastic collision the kinetic energy loss in the forward direction is proportional to $\cos^2(2\varphi_{max})$, we opted, for a number of reasons (see Results and discussion), to approximate the energy loss with a simple cosine function. Thus, the two energy levels can be formalized as:

$$TE_{max} = \tfrac{1}{2}m\bar{s}^2$$

and $$TE_{min} = TE_{max}\cos(2\varphi_{max}).$$ (A22)

Only the kinetic component is represented in the equation for total energy because, despite the transformations between potential and kinetic energies during the inverted pendulum oscillation, it constitutes the maximum energy of the system in the forward direction.

The rate of external mechanical work is:

$$\dot{W}_{ext} = 2f\Delta TE$$

$$= 2\tfrac{1}{2}fm\bar{s}^2[1 - \cos(2\varphi_{max})]$$ (A23)

(as for $\dot{W}_{int}$ the multiplying factor 2 accounts for the two energy rises during one stride).

By substituting equation A17 into equation A23 and simplifying, the equation for $\dot{W}_{ext}$ becomes:

$$\dot{W}_{ext} = \frac{m\bar{s}^4}{8fr^2}. \tag{A24}$$

Expressing f in Hz and $\bar{s}$ in ms^{-1}, and allowing for standard anthropometric values, equation A24 estimates the external work rate in watts as:

$$\dot{W}_{ext} = 10.171\frac{\bar{s}^4}{f}. \tag{A25}$$

Stride frequency for minimum $\dot{W}_{tot}$

Adding equations A20 and A24 (or equations A21 and A25) gives the total mechanical work rate of walking:

$$\dot{W}_{tot} = \dot{W}_{int} + \dot{W}_{ext}$$

$$= \bar{s}^2 \left\{ \frac{f\pi^2}{2}[(a^2 + g_0{}^2)(m_L + b^2 m_U)] + \frac{m\bar{s}^2}{8fr^2} \right\}. \tag{A26}$$

The technique for obtaining the stride frequency at which the work rate is minimal can be summarized as follows:

$$\frac{\partial \dot{W}_{tot}}{\partial f} = 0 \rightarrow f_{opt} = \frac{\bar{s}}{2\pi r}\sqrt{\frac{m}{(a^2 + g_0{}^2)(m_L + b^2 m_U)}}, \tag{A27}$$

i.e. take the partial derivative of equation A26 with respect to f, equate it to zero, and express the result as a function of $\bar{s}$. The optimal stride frequency (f_{opt}) for the standard man, when parameter units are expressed as indicated previously, is:

$$f_{opt} = 0.685\bar{s}. \tag{A28}$$

List of abbreviations

The values in parentheses have been used in this paper to predict the natural stride frequency of walking of a 'standard man' (height=1.75 m).

a	proximal distance of lower limb centre of mass as a fraction of r (44.7 %)
b	upper limb length as a fraction of r (83.2 %)
f	stride frequency (Hz)
f_{opt}	stride frequency for minimum rate of total work (Hz)
g_L, g_U	lower and upper limb radii of gyration (limb length fraction) (32.6, 36.8 %)
g_0	mean radius of gyration (lower limb length fraction)
I_L, I_U	lower and upper limb moment of inertia

m	body mass (70 kg)
m_L, m_U	mass of lower and upper limbs (11.27 and 3.50 kg)
r	lower limb length (0.928 m)
RKE_L, RKE_U	rotational kinetic energy of lower and upper limbs
$\bar{s}$	average progression speed (m s^{-1})
$s_L(t), s_U(t)$	movement of lower and upper extremities
$s_{Lcom}(t), s_{Ucom}(t)$	movement of lower and upper limb centre of mass
$\dot{s}_{Lcom}(t), \dot{s}_{Ucom}(t)$	speed of lower and upper limb centre of mass
TE_{max}	total energy of the body centre of mass: maximum level
TE_{min}	total energy of the body centre of mass: minimum level
$TKE_{Lcom}(t), TKE_{Ucom}(t)$	translational kinetic energy of lower and upper limbs
$\dot{W}_{int}$	rate of internal mechanical work (W kg^{-1})
$\dot{W}_{ext}$	rate of external mechanical work (W kg^{-1})
$\dot{W}_{tot}$	rate of total mechanical work (W kg^{-1})
ΔRKE_t	total range of rotational kinetic energy
ΔTE	total energy range of the body centre of mass
$\Delta TKE_{Lcom}, \Delta TKE_{Ucom}$	TKE range of lower and upper limbs
ΔTKE_t	total range of translational kinetic energy
ρ_L, ρ_U	lower and upper limb radii of gyration (about the centre of mass)
φ	angle between the vertical line and limbs (rad)
$\varphi(t)$	φ at time t (rad)
φ_{max}	maximum φ during the stride (rad)
$\dot{\varphi}(t)$	angular speed
ω	swing frequency

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